Maa Omwati Degree College (Hassanpur)

Examination Notes(2025-26)

Subject- ORDINARY DIFFERENTIAL EQUATION

Class-B.sc 3RD (SEM)

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Date

Function

Algebraic func

→ Those function which involve algebraic operation (+, −, ×, ÷)

- Polynomial function
- Rational function
- Power function ($f(x) = ax^{\alpha}$) $\alpha \in Q, Z$

Transcendental func.

Those function which are not algebraic

- Trignometry
- Inverse trignometry
- Logarithmic
- Exponential
- Hyperbolic

Independent and dependent variable:→

→ A variable whose value is assigned is known as Independent variable whereas variable whose value depen on other variable is known as dependent variable.

$y = f(x)$

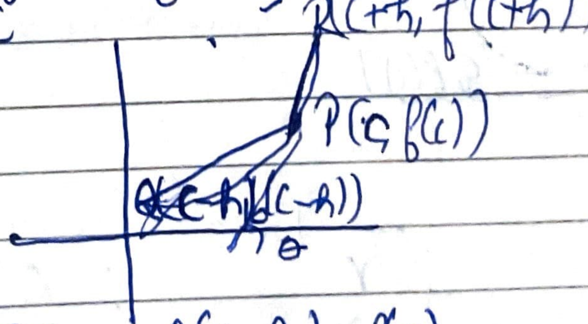
x → Independent variable

y → Dependent variable

Geometrical Interpretation of Diffrenticibility

Let f be a diff. function at $x = c$

[i.e. $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$]



Slope of chord PQ = $\frac{f(c-h) - f(c)}{-h}$

Slope of chord PR = $\frac{f(c+h) - f(c)}{h}$

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$$\lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} = \lim_{h\to 0} \text{slope of chord } PQ$$

$$= \lim_{Q\to P} \text{slope of chord } PQ$$

$$= \text{slope of tangent at } P$$

Now similarly

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \text{slope of tangent at } P$$

$\therefore f(x)$ is diff. at $x=c$ iff there exist a unique tangent at $x=c$

Further $f'(c) =$ slope of tangent at $x=c$

$= \tan\theta$ [Anticlockwise direction]

Ex: $f(x)=|x|$

At $x=0$, we have infinite tangent

$\therefore$ Not diffrentiable at $x=0$

Eg $f(x) = |\sin x|$

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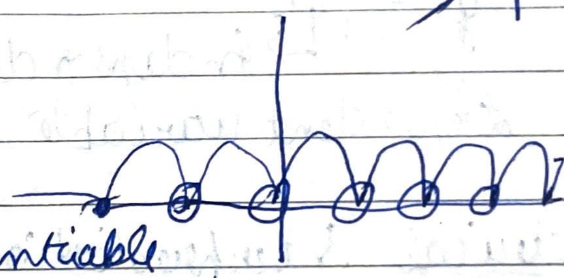
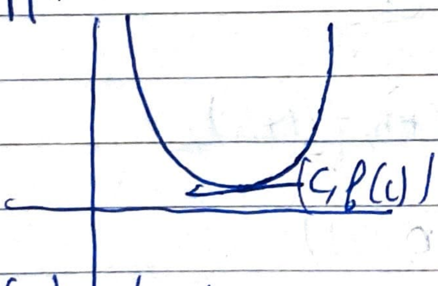
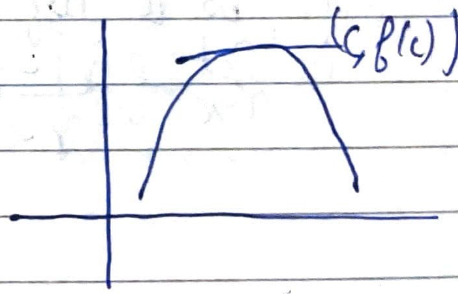


$f(x)=|\sin x|$ is not diffrentiable at $n\pi$.

Eg:





Eg: $f(x) = |x|$

$$f(x) = \begin{cases} x & ; x>0 \\ -x & ; x<0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & ; x>0 \\ -1 & x<0. \end{cases}$$

RHD $\neq$ LHD at $x=0$

Therefore function is not diff at $x=0$

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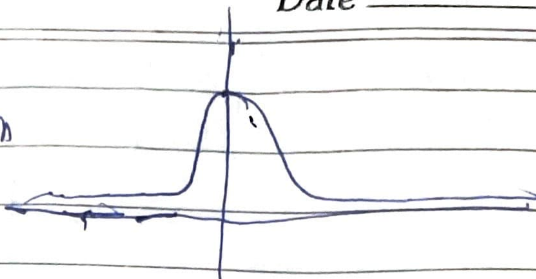
$f'(x) < 0 \quad \forall x \geq 0$

$f(x)$ is decreasing function

Range $f = (0,1]$



★ Ordinary diff. equation: ~~An equation involve~~

Let y be any real valued function defined on I.

Any relation of the form $F(x, y, y', y'', \ldots, y^{n}) = 0$ is defined as n^{th} order ordinary differential equation and $n \geq 1$

where $F : I \times \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \to \mathbb{R}$ is smooth function [continuously differentiable function] and "$\partial F / \partial y^{n} \neq 0$"

Remark:

- Domain of y is subset of $\mathbb{R}$. Hence, derivative involved are ordinary derivative.
- Assume for $n = 1$

$F(x, y, y') = y$

Consider $F(x, y, y') = 0$

$\Rightarrow y = 0$

Not an differential equation.

$F(x, y, y') = 0$ should not be independent of y'

★ Order of ODE: Highest order derivative involved in the diff. equation is defined as order of ODE.

Eg: $\left(\frac{dy}{dx}\right)^3 = \sin x$ ∴ Order-1

Eg: $(y')^3 + e^{y''} = 0$ ∴ Order ⇒ 2

★ Degree of ODE: Highest power of the highest order derivative is defined as degree of ODE provided all ~~the~~ derivative are in natural power.

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(Or) Derivative are free from radical, fraction

Note:-

$$f_1(x,y)\left(\frac{d^{n_1}y}{dx^{n_1}}\right)^{m_1}+f_2(x,y)\left(\frac{d^{n_2}y}{dx^{n_2}}\right)^{m_2}+\dots+f(x,y)\left(\frac{dy}{dx}\right)^{m_r}=0$$

i.e. all the derivative are expressed in polynomial

Eg- $\frac{d^2y}{dx^2}=\sqrt{y-\frac{dy}{dx}} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)-y=0$

Degree-2

Eg: $(y')^3+e^{y''}=0 \Rightarrow e^{y''}=-(y')^3$

$y''-3\log y'=0$

Order = 2, Degree = not defined (undefined).

Eg: $(y')^3+e^{y}=0$

Order = 1 Degree = 3.

Formation of ODE:- Consider $F(x,y,c_1,c_2,\dots c_n)=0$ where $c_1,c_2,\dots c_n$ are constant

- We will find first n-successive derivative
- We will use this n derivative and given equation to eliminate $c_1,c_2,\dots c_n$
- We will get required diff equation

Remark:- The order of the formed differential equation is equal to the number of constant present in the given equation.

Eg: Find the differential equation of the family of straight line.

$y=mx+c$

where

i) m is constant

ii) c is constant

iii) m and c are constant

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$\frac{dy}{dx} = m \Rightarrow y = \cancel{y} y' x + c$

ii) c is constant

$\frac{dy}{dx} = m$

(iii) m and c are constant

$\frac{dy}{dx} = m \Rightarrow \frac{d^2y}{dx^2} = 0$

Eg 2. $y = ae^{4x} + be^{-x}$ where a and b are constan

$\frac{dy}{dx} = 4ae^{4x} - be^{-x}$

$\frac{d^2y}{dx^2} = 16ae^{4x} + be^{-x}$

$20ae^{4x} = \frac{dy}{dx} + \frac{d^2y}{dx^2}$

Sol

$a = \frac{e^{-4x}}{20}\left(\frac{d^2y}{dx^2} + \frac{dy}{dx}\right)$

$\frac{dy}{dx} = \frac{1}{5}\left(\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) = be^{-x}$

$be^{-x} = \frac{1}{5}\frac{d^2y}{dx^2} - \frac{4}{5}\frac{dy}{dx}$

$b = \frac{e^{x}}{5}\left(\frac{d^2y}{dx^2} - 4\frac{dy}{dx}\right)$

$y = \frac{1}{20}\left(\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) + \frac{1}{5}\left(\frac{d^2y}{dx^2} - 4\frac{dy}{dx}\right)$

$20y = \frac{d^2y}{dx^2} + \frac{dy}{dx} + 4\frac{d^2y}{dx^2} - 16\frac{dy}{dx}$

$\frac{3 \pm 5}{2} =$

$5\frac{d^2y}{dx^2} + \cancel{20y = 0} - 15\frac{dy}{dx} - 20y = 0$

$\frac{3 \pm \sqrt{9+16}}{2}$

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$

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Ex: $(x-a)^2+(y-b)^2=r^2$ where a, b and r are constts

$2(x-a)+2(y-b)\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{(x-a)}{(y-b)}$

$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$

$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$ ---(1)

$\Rightarrow (y-b)\frac{d^3y}{dx^3}+y'y''+2y'y''=0$

$(y-b)y'''+3y'y''=0$ ---(2)

$\cancel{(y-b)y'''=-3y'y''}$ ---(2)

$y+\frac{3y'y''}{y'''}=b$

From (1) and (2)

$\boxed{[1+(y')^2]y'''-3y'(y'')^2=0}$

Lect 3 [24-07-25]

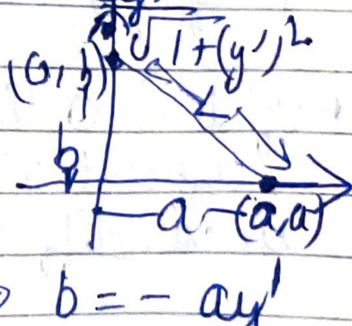
Que:- The differential equation which represent family of straight line which have an intercept of constant length 'L' between coordinate axis.

a) $y'-y=\frac{Ly'}{\sqrt{1+(y')^2}}$

b) $xy'-y=\frac{Ly}{\sqrt{1+(y')^2}}$

c) $xy'+y=\frac{Ly'}{\sqrt{1+(y')^2}}$

d) $y'+y=\frac{Ly'}{\sqrt{1+(y')^2}}$



$y=mx+c$ $\quad \frac{x}{a}+\frac{y}{b}=1$

$\frac{dy}{dx}=m$ $\quad bx+ay=ab$

$b+a\frac{dy}{dx}=0 \Rightarrow a\frac{dy}{dx}=-b \Rightarrow b=-ay'$

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$\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{a}+\frac{y}{-ay'}=1$

$\frac{1}{a}\left(x-\frac{y}{y'}\right)=1 \Rightarrow xy'-y=ay'$

By PGT $a^2+b^2=L^2$

$a^2+a^2(y')^2=L^2$

$L=a\sqrt{1+(y')^2} \Rightarrow a=\frac{L}{\sqrt{1+(y')^2}}$

$xy'-y=\frac{Ly'}{\sqrt{1+(y')^2}}$. (First quadrant)

Ques ☆ Linear differential equation :⇒ An n^{th} order diff. equation is said to be linear if it is of the form,

$a_0(x)y^n+a_1(x)y^{n-1}+\dots+a_n(x)y=b(x)$ where $a_0(x)\neq 0$

$\boxed{\text{Linear} \Leftrightarrow \text{Order}(1)}$

Observation:-

$T: C^2(I)\to C(I)$.

$T(y)=a_0(x)y''+a_1(x)y'+a_2(x)y$.

$T(y_1+y_2)=a_0(x)(y_1+y_2)''+a_1(x)(y_1+y_2)'+a_2(x)(y_1+y_2)$

$=a_0(x)y_1''+a_1(x)y_1'+a_2(x)y_1+a_0(x)y_2''+a_1(x)y_2'+a_2(x)y_2$

$=T(y_1(x))+T(y_2(x))$

$\boxed{T(y_1+y_2)=T(y_1)+T(y_2)}$

- If $b(x)=0$ then it is known as homogenous linear differential equation else it is non homogenous linear diff. equation.

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- If $a_0(x), a_1(x), \ldots a_n(x)$ are constant then it is known as linear differential equation with constant coefficient else it is known as linear diff. equation with variable coefficient.

Eg:- $\left|\frac{dy}{dx}\right| + |y| = 2$. Non linear

Eg:- $yy' + \log(xy) = 0$ Non linear

Eg:- $y'' + \sqrt{y} = \sin x$ Non linear

Que:- let $f(x, y, y', \ldots y^n) = 0$ be any n^{th} order diff. eq.

Choose incorrect

$yy' + \log(xy) = 0$

(a) If degree of $f = 1$ then $f = 0$ is linear diff. equation.

(b) If $f = 0$ is linear diff. equation then degree of $f = 1$.

(c) If degree of $f > 1$ then $f = 0$ is linear diff. equation

(d) If $f = 0$ is non linear then degree of f is greater than 1

$yy' + \log(xy) = 0$.

Envelope :- Envelope of the system of surfaces touches every member of the system.

Let $f(x, y, z, a) = 0$ be the given system of surface.

Consider

$f(x, y, a + \Delta a) = 0$ be the another member of the system

For $\Delta a \neq 0$

$f(x, y, z, a) = 0$

$$\frac{f(x, y, z, a + \Delta a) - f(x, y, z, a)}{\Delta a} = 0$$

On taking $\Delta a \to 0$

$f(x, y, z, a) = 0$ — (1)

$f_a(x, y, z, a) = 0$ — (2)

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using (1) and (2):
we eliminate 'a' and we get envelope of the system

Eg: $x^2+y^2+(z-a)^2=1$.

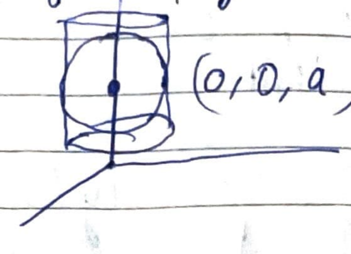
$f(x,y,z,a) = x^2+y^2+(z-a)^2-1=0$

$f_a(x,y,z,a) = 2(z-a)=0$

$z=a$.

$x^2+y^2+(a-a)^2-1=0$

$\boxed{x^2+y^2=1}$ Equation of ~~circle~~ cylinder in 3 D coordinate



★ Solution of a diff. equation :-

Given $F(x, y, y', y'', \dots, y^n) = 0$ — (*)

Any n^{th} order ~~derivati~~ differential equation.

A function y is said to be solution of differential if

i) $y \in C^n(I)$

ii) y satisfies (*) $\forall x \in I$.

Eg: $y'-y=0$

$y'=y$

$\frac{dy}{y} = dx \Rightarrow \log y = x+c$

$y = ce^x$

$y=e^x$, $y=0$ are the solution

Eg: $y''+y=0$

$y=\cos x$, $y=\sin x$, $y=0$ are the solution

★ Types of ~~solutio~~ solution:

i) General solution :- The solution of the differential equation in which no of constant is equal to the order of the diff equation is known as general solution.

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ii) **Particular solution**: The solution of the differential equation which is obtained from general solution by assigning the value of the constant is known as particular solution

iii) **Singular solution**:- The solution of diff. equation which can not be obtained from general solution is known as singular solution

⇒ Singular solution is the envelope of the general solution.

Eg: $\frac{dy}{dx} = (y-3)^2$ ⇒ $y-3 = t$, $dy = dt$

$\frac{dy}{(y-3)^2} = dx$

$\frac{dt}{t^2} = dx \Rightarrow = -\frac{1}{t} = x + c$

$-\frac{1}{(y-3)} = x + c$ ⇒ $\boxed{y = 3 - \frac{1}{x+c}}$

$\boxed{x(y-3) + c(y-3) + 1 = 0}$ General solution

Now, $\boxed{y = 3 - \frac{1}{x}}$ ∴ $c = 0$ Particular solution

$\boxed{y = 3}$ Singular solution

$\frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2}$

$y = 2$, $A = \frac{1}{4}$

$y = -2$, $B = -1/4$

Eg: $\frac{dy}{dx} = y^2 - 4$ ⇒ $\frac{dy}{(y-2)(y+2)} = dx$

$\frac{1}{4}\log|y-2| - \frac{1}{4}\log|y+2| = x + c$

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$\frac{1}{2} =$

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$\frac{1}{4}\log\left|\frac{y-2}{y+2}\right| = x + C.$

$\log\left|\frac{y-2}{y+2}\right| = 4x+4C.$

$\frac{y-2}{y+2} = e^{4x+4C}.$

$\frac{y-2}{y+2} = Ae^{4x}$ $\Rightarrow y-2 = Ae^{4x}y + 2Ae^{4x}$

$y(1-Ae^{4x}) = \cancel{2y+2}\ 2Ae^{4x}+2$

$y = \frac{2(1+Ae^{4x})}{1-Ae^{4x}}$ // General

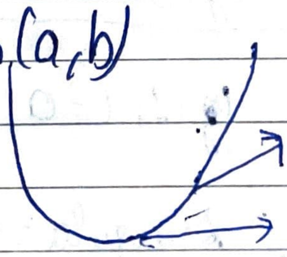
Put $A=0$, $y=2$. // Particular solution

$\boxed{y=-2}$ // Singular solution

Remark: If f is continuous and $f(a)f(b)<0$ $\exists\, c\in(a,b)$ such that $f(c)=0$

Concave up / Concave down $\Rightarrow$ Assume f has continuous 2nd order derivative in (a,b).

(convex) Concave up:- A function $f(x)$ is said to be concave up in (a,b) $\Leftrightarrow$ $f'(x)$ is increasing in (a,b)



- Tangent at every point lies below the curve
- As x increases, tangent at $(x, f(x))$ increases in anticlockwise direction

$f'>0$

$f''(x)>0 \ \forall\, x\in(a,b)$

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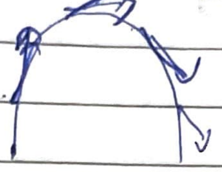
Concave down: A function $f(x)$ is said to concave down in (a,b) $\Leftrightarrow$ $f'(x)$ is ~~decreasi~~ decreasing in (a,b)



- Tangent at every point lies above the curve.
- As x increase; tangent at $(x, f(x))$ decreases in ~~anti~~clockwise direction.
- $f''(x) < 0 \quad \forall\ x \in (a,b)$

★ **Point of inflection** $\Rightarrow$ A point $(c, f(c))$; c lies between $a < c < b$ is point of inflection if at $(c, f(c))$ concavity changes i.e concave up to concave down or concave down to concave up.

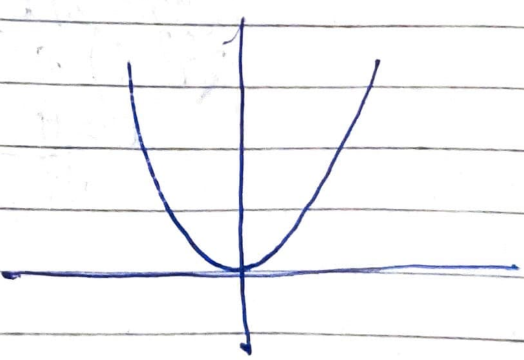
Since $(c, f(c))$ is point of inflection
$f''(c-h),\ f''(c+h) < 0$
$$\boxed{f''(c) = 0}$$

Remark:

- If $f''(c)$ does not exist
But $f''(c-h),\ f''(c+h) < 0$
then $(c, f(c))$ is point of inflection

- If $f''(c) = 0$ ~~then~~
then $(c, f(c))$ is ~~need~~ need not to be point of inflection

Eg $f(x) = x^4$
$f'(x) = 4x^3$
$f''(x) = 12x^2$
$f''(x) = 0$
$\Rightarrow x = 0$



$y(0) = 0$

$0 = 0 + 2c$

$C = 0$

so, $\boxed{y = \left(\frac{2x}{3}\right)^{3/2}}$ & $\boxed{y = 0}$

- The existence and uniqueness theorem:

Hypothesis:-

Consider the diff. eqn:

$$\frac{dy}{dx} = f(x, y)$$

- f is continous of x and y in the domain $D(x_0, y_0) \to$ Pt. in D.

Conclusion:-

Then there exist a unique sol. of the diff. eq. defined on some domain $|x - x_0| \le h$ where h is sufficiently small and $\phi(x_0) = y_0$

Ex:- Consider the intial value problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(1) = 3.$$

$f(x, y) = x^2 + y^2 \to$ contin. on $R \times R$.

$\frac{\partial f}{\partial y} = 2y \to$ cont. on R.

Partial derivative

$\therefore$ There exist a unique sol.

Ex:- Solve the intial value problem:

$$\frac{dy}{dx} = -6xy.$$

i) $y(0) = 7$ & (ii) $y(0) = -4$.

Sol:- $\frac{dy}{dx} = -6xy \Rightarrow \int \frac{dy}{y} = -\int 6x\,dx$

$\log y = -3x^2 + C$

i) As $y(0) = 7$, Put $x = 0, y = 7$

$\log 7 = -3(0)^2 + C$

$C = \log 7$

$\log y = -3x^2 + \log 7$

$\log\left(\frac{y}{7}\right) = -3x^2$

$\frac{y}{7} = e^{-3x^2} \Rightarrow \boxed{y = 7e^{-3x^2}}$

ii) As $y(0) = -4$, so $x = 0, y = -4$

$\log|-4| = -3(0)^2 + C$

$C = \log 4$

$\log|-4| = -3x^2 + \log 4$

$\log\left|\frac{y}{4}\right| = -3x^2$

$y = -4e^{-3x^2}$

Exact diff. eqn:- Consider the eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$\frac{\partial M}{\partial Y} = \frac{\partial N}{\partial X}$ $\therefore$ Necessary condition

then the diff. eqn. is said to be exact.

Let F be a function of two variable such th has continuous first partial der. in a dom

Then the total der. of F is given

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial Y}dy.$$

Ex- $y^2dx + 2xy\,dy = 0$

$\frac{\partial M}{\partial Y} = 2y$ $\quad$ $\frac{\partial N}{\partial X} = 2y.$

Hence $\frac{\partial M}{\partial Y} = \frac{\partial N}{\partial X}$ (exact).

Ex: $y\,dx + 2x\,dy = 0$

$M(x,y) = y \qquad N(x,y) = 2x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Not exact.

- Solution of an exact diff. eqn.

Suppose that the diff. eqn.

$M(x,y)\,dx + N(x,y)\,dy = 0$

is exact in some domain (rectangular) D.

Then a family of sol. of this diff. eqn is given

$f(x,y) = C$

where f is a function such that

$\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$

for all $(x,y) \in D$

Ex: Solve the diff. equation:

$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$

$M = 3x^2 + 4xy, \qquad N = 2x^2 + 2y$

$\frac{\partial M}{\partial y} = 4x \qquad \frac{\partial N}{\partial x} = 4x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact

★ In these respect we diff. it and all other are constant

जिसके respect मे करते है सिर्फ उसी Term को करेंगे बाकी const

We have to find $f(x,y)$ such that

$\frac{\partial F}{\partial x} = 3x^2 + 4xy$

Integrating both side w.r.t to 'x'

$F(x,y) = x^3 + 2x^2y + \theta(y) \quad -(1)$

Diff. w.r.t to 'y'

$\frac{dF(x,y)}{dx} = 3x^2 + 2x^2 y + \theta' y.$

$2x^2 + 2y = 2x^2 + \theta'(y)$

$\theta'(y) = 2y^2$

Integration both side w.r.t to 'y'

$\theta(y) = 2y^2 + C_1$

Put this in eq. 1

$F(x,y) = x^3 + 2x^2y + y^2 + C_1$

$F(x,y) = C$

$C - C_1 = x^3 + 2x^2y + y^2.$

Put $C - C_1 = \bar{C}$

$x^3 + 2x^2y + y^2 = \bar{C}$

2. $(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0$

$\frac{\partial M}{\partial y} = -2x\sin y + 3x^2 \qquad \frac{\partial N}{\partial x} = 3x^2 - 2x\sin y.$

Exact

To find: $F(x,y)$

$\frac{\partial F(x,y)}{\partial x} = 2x\cos y + 3x^2y$

Int. both side w.r.t to 'x'

$F(x,y) = x^2\cos y + x^3 y + \theta(y). \quad -(1)$

Diff. w.r.t to 'x'

$\frac{dF(x,y)}{dx} = \cancel{2x}\,x^2\cos y + 3x^2 y + \theta'(y).$

$x^3 - x^2\sin y - y = -x^2\sin y + x^3 + \theta'(y)$

$\theta'(y) = -y$

Int. w.r.t to 'y'

$\theta(y) = -\frac{y^2}{2} + C_1$

Put this in eq. 1

$F(x,y) = x^2\cos y + x^3 y - \frac{y^2}{2} + C_1$

$\therefore F(x,y) = C$

$x^2\cos y + x^3 y - \frac{y^2}{2} = C - C_1$

$x^2\cos y + x^3 y - \frac{y^2}{2} = \bar{C}$

given $y(0) = 2$

$-4 = 2\bar{C}$

$C = -2$

Required eqn $= x^2\cos y + x^3 y - \frac{y^2}{2} + 2 = 0$

Note:- $M(x,y)\,dx + N(x,y)\,dy = 0$

Exact

$\int M(x,y)\,dx + \int$ (Terms not containing x in $N(x,y)$) dy.

3. $(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$

$\frac{\partial M}{\partial y} = 6x - 3y^2$ $\qquad \frac{\partial N}{\partial x} = 6x - 3y^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ($\therefore$ exact)

To find: $F(x,y)$

$\frac{\partial F(x,y)}{\partial x} = 6xy - y^3$

Int. w.r.t. to x'

$F(x,y) = 3x^2y - xy^3 + \theta(y) \qquad -(1)$

Diff. w.r.t. to y'

$\frac{dF(x,y)}{dx} = 3x^2 - 3xy^2 + \theta'(y)$

$4y + \cancel{3x^2} - \cancel{3xy^2} = \cancel{3x^2} - \cancel{3xy^2} + \theta'(y)$

$\theta'(y) = 4y$

Int. w.r.t to y'

$\theta(y) = 2y^2 + C_1$

Put this in eq. 1

$F(x,y) = 3x^2y - xy^3 + 2y^2 + C_1$

$F(x,y) = C$

$3x^2y - xy^3 + 2y^2 = C - C_1$

$3x^2y - xy^3 + 2y^2 = \bar{C}$

☆ Integration factor

$M(x,y)\,dx + N(x,y)\,dy = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (Not exact)

but $u(x,y)\,M(x,y)\,dx + u(x,y)\,N(x,y)\,dy = 0$ is exact.

Then $u(x,y)$ is called as integrating factor of diff. eqn.

☆ Separable equation and equations reducible to this form:

An eqn of the form

$F(x)\,G(y)\,dx + H(x)\,K(y)\,dy = 0$

is a separable diffrential equation.

1. $(x-4)\,y^4\,dx - x^3(y^2-3)\,dy = 0$

$(x-4)\,y^4\,dx = x^3(y^2-3)\,dy$.

$\frac{(y^2-3)\,dy}{y^4} = \frac{(x-4)\,dx}{x^3}$

$\Rightarrow \left(\frac{1}{y^2} - \frac{3}{y^4}\right)dy = \left(\frac{1}{x^2} - \frac{4}{x^3}\right)dx$

Integrating both side, we get-

$$-\frac{1}{y}+\frac{1}{y^3}=-\frac{1}{x}+\frac{2}{x^2}+C$$

☆ Homogenous diff eqn-

$$(x^2-3y^2)dy+2xy\,dx=0$$

$$\frac{dy}{dx}=\frac{2xy}{3y^2-x^2} \quad -(1)$$

~~Put v=yx~~ Put $y=vx$

$$\cancel{\frac{dv}{dx}+}\ \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$\frac{dy}{dx}$ = 2 Put this in eq1

$$v+x\frac{dv}{dx}=\frac{2vx\cdot x^2}{3v^2x^2-x^2}$$

$$v+x\frac{dv}{dx}=\frac{2v}{3v^2-1}$$

$$x\frac{dv}{dx}=\frac{2v}{3v^2-1}-v$$

$$x\frac{dv}{dx}=\frac{2v-3v^3+v}{3v^2-1}$$

$$x\frac{dv}{dx}=\frac{3v-3v^3}{3v^2-1}$$

$$\frac{3v^2-1}{3v-3v^3}dv=\frac{dx}{x}$$

Int. both side

$$\int\left(\frac{3v^2-1}{3v-3v^3}\right)dv=\int\frac{dx}{x}$$

$3v-3v^3\to t$

$(3-9v^2)dv\to dt$

$-3(3v^2-1)dv\to dt$

$$\to\frac{dt}{-3}.$$

$$-\frac{1}{3}\int\frac{dt}{t}=\int\frac{dx}{x}$$

$$\log\left|\frac{1}{t}\right|=\log|cx^3|$$

$$\log\left|\frac{1}{v}\right|=\log|cx^3|$$

Taking antilog both side.

$$\frac{1}{v}=cx^3.$$

$$\frac{x}{y}=cx^2$$

$$\frac{1}{x^2y}=c.$$

☆ $\frac{dy}{dx}=f(x,y)$

$f(tx,ty)=t^n f(x,y)\to$ homogenous of degree n.

1. $(y+\sqrt{x^2+y^2})dx-x\,dy=0.$

$M(x,y)=y+\sqrt{x^2+y^2}$ $\quad N(x,y)=-x.$

$$M(tx,ty)=ty+\sqrt{t^2x^2+t^2y^2}$$

$$=t(y+\sqrt{x^2+y^2})$$

$$=tM(x,y).$$

$N(tx,ty)=-tx=tN(x,y)$

∴ Homogenous diff. eq of degree 1.

- If $M(x,y)dx+N(x,y)dy=0$ is a homogenous diff then the change of variable $y=vx$ transforms the given diff eqn. has a separable eqn.

Linear diff. eqn: A first order ordinary diff. eq is linearly in the dependent variable y and independent variable x of it is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Integrating factor $= e^{\int P(x)dx} = I.f$

Then the solution is given by

$$y(I.F) = \int I.F \times Q(x)dx + C$$

Ex: $(x^2+1)\frac{dy}{dx} + 4xy = x$

$$\frac{dy}{dx} + \frac{4xy}{x^2+1} = \frac{x}{x^2+1}$$

$$I.F = e^{\int \frac{2\cdot 2x}{x^2+1}dx} = e^{2\log(x^2+1)} = (x^2+1)^2$$

$$y(x^2+1)^2 = \int (x^2+1)^2 \times \frac{x}{(x^2+1)}dx + C$$

$$y(x^2+1)^2 = \int x^3 + x$$

$$y(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + C$$

As $y(2) = 1$

Put $x=2$ and $y=1$ we get

$1 \times (25) = 4 + 2 + C$

$25 = 6 + C$

$C = 19$

$$y(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + 19$$

$$y = \frac{1}{(x^2+1)^2}\left[\frac{x^4}{4} + \frac{x^2}{2} + 19\right]$$

Implicit function: जिसमें x से separate नहीं है।

Explicit function: जिसमें y से separate होता है।

☆ Bernoulli equation

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$ is called Bernoulli equation

Ex. If $n=0 \Rightarrow$ linear

If $n=1 \Rightarrow \frac{dy}{dx} + P(x)y = Q(x)y$

$$\frac{dy}{dx} = [Q(x) - P(x)]y$$

OR $$\frac{dy}{dx} + [P(x) - Q(x)]y = 0$$

Theorem:- Suppose that $n \neq 0$ or 1 the transformation $v = y^{1-n}$ reduces the Bernoulli's equation

$\frac{dy}{dx} + P(x)y = Q(x)y^n$ to a linear eqn in v.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^{n-1}} = Q(x)$$

~~let $v = y$~~ $y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$

let $v = y^{1-n}$

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} \times \frac{1}{(1-n)}$$

$$\frac{1}{(1-n)}\frac{dv}{dx} + P(x)v = Q(x)$$

$$\frac{dv}{dx} + P(x)(1-n)v = Q(x)(1-n)$$

$\frac{dy}{dx} + P_1(x)y = Q(x)$

Eg: $\frac{dy}{dx} + xy = xy^3$

$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x$

$v = \frac{1}{y^2}$

$\frac{dv}{dx} = \frac{-2}{y^3}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$

$-\frac{1}{2}\frac{dv}{dx} + v = x$

$\frac{dv}{dx} - 2v = -2x$

$I \cdot F = e^{\int -2dx} = e^{-2x}$

$ve^{-2x} = \int e^{-2x}(-2x) + c$

$ve^{-2x} = -2\left[\frac{xe^{-2x}}{-2} + \frac{1}{2}\int e^{-2x}\,dx\right] + c$

$= -2\left[\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}\right] + c$

$v = x + \frac{1}{2} + ce^{2x}$

Put $v = 1/y^2$

$\frac{1}{y^2} = \frac{2x + 1 + 2ce^{2x}}{2}$

$\boxed{y^2 = \frac{2}{2x + 1 + 2ce^{2x}}}$ (explicit)

Ex: $\frac{dy}{dx} + y = f(x)$ and $y(0) = 1$ $f(x) = \begin{cases} e^{-x} & 0 \le x \le 2 \\ e^{-2} & x \ge 2 \end{cases}$

$\frac{dy}{dx} + y = e^{-x}$

$I \cdot F = ye^{x} = \int e^{x} \cdot e^{-x}\,dx + c$

$ye^{x} = x + c$

$y(0) = 1 \Rightarrow c = 1$

$ye^{x} = x + 1$

$y = e^{-x}(x+1)$

Now, if $x \ge 2$.

$\frac{dy}{dx} + y = e^{-2}$

$I \cdot F = e^{x}$

Soln is $ye^{x} = \int e^{-2} \cdot e^{x}\,dx$

$ye^{x} = e^{x}e^{-2} + c$

$y = e^{-x}(x+1)$.

$y(2) = e^{-2}(3) = 3e^{-2}$

Use there $y(2) = 3e^{-2}$

$3e^{-2} \cdot e^{2} = e^{0} + c$

$c = 2$

$ye^{x} = e^{x} \cdot e^{-2} + 2$

$y = e^{-2} + 2e^{-x}$

$y = \begin{cases} e^{-x}(x+1) & 0 \le x < 2 \\ e^{-2} + 2e^{-x} & x \le 2 \end{cases}$

★ Special integrating factor.

$M(x,y)\,dx + N(x,y)\,dy = 0$

Not exact $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$u(x,y)M(x,y)\,dx + u(x,y)N(x,y)\,dy = 0$

$M_1(x,y) + N_1(x,y) = 0$

exact $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

Try to make an inexact equation exact

$\frac{1}{N(x,y)}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ has only variable x

I.F. $= e^{\int N(x,y)\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]dx}$

$\frac{1}{M(x,y)}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial x}\right]$ has only variable.

I.F. $= e^{\int M(x,y)\left[\frac{\partial M}{\partial y}-\frac{\partial M}{\partial x}\right]dy}$

Ex $(2x^2+y)\,dx+(x^2y-x)\,dy=0$

$\frac{\partial M}{\partial y}=1,\quad \frac{\partial N}{\partial x}=2xy-1$

$\frac{1}{x^2y-x}\left[2xy-2\right]=\frac{2\left[1-xy\right]}{x(xy-1)}=\frac{-2}{x}$

I.F. $= e^{\int\frac{-2}{x}dx}=e^{-2\int\frac{dx}{x}}=e^{\log x^{-2}}=\frac{1}{x^2}$

Multiply 1 by x^2

$\left(2+\frac{y}{x^2}\right)dx+\left(y-\frac{1}{x}\right)dy=0$

$\frac{\partial M_1}{\partial y}=\frac{1}{x^2},\quad \frac{\partial N}{\partial x}=\frac{1}{x^2}$.

Now we find $F(x,y)$.

$\frac{\partial F}{\partial x}=\left(2+\frac{y}{x^2}\right)$

Int w.r.t to 'x'

$F(x,y)=2x-\frac{y}{x}+\theta(y)\quad -(1)$

Diff w.r.t to 'y'

$\frac{dF(x,y)}{dy}=\frac{y}{x^2}+\theta'(y)$.

$y-\frac{1}{x}=-\frac{1}{x}+\theta'(y)$

$\theta'(y)=y$

Int w.r.t to 'y'

$\theta(y)=\frac{y^2}{2}+C_1$

Put in eq 1

$F(x,y)=2x-\frac{y}{x}+\frac{y^2}{2}+C_1$

$F(x,y)=C$.

$2x-\frac{y}{x}+\frac{y^2}{2}=C-C_1$

$\boxed{2x-\frac{y}{x}+\frac{y^2}{2}=\bar{C}}\qquad\boxed{\therefore\ \bar{C}=C-C_1}$

A special transformation.

Theorem: Consider the diff. eqn.

$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$

where c_1, a_1, b_1, b_2, c_1 & c_2 are constant.

case 1 $=\frac{a_2}{a_1}\neq\frac{b_2}{b_1}$

Let $x=X+h$ and $y=Y+K$ where (h,K) is sol of system.

$a_1h+b_2K+c_1=0$ (solve to find h,K)

$a_2h+b_2K+c_2=0$

$dx=dX$ & $dy=dY$

Eqn (1) reduce to

$(a_1X+b_1Y)dX+(a_2X+b_2Y)dy=0$

$\frac{dy}{dX}=\frac{-(a_1X+b_1Y)}{(a_2X+b_2Y)}$.

Homogeneous diff eqn is X and Y

$X=x-h$

$Y=y-K$.

Case 2: $\frac{a_2}{a_1}=\frac{b_2}{b_1}\Rightarrow a_2=\lambda a_1$ & $b_2=\lambda b_1$

$(\underbrace{a_1x+b_1y+c_1}_{z})dx+(\underbrace{a_2x+b_2y+c_2}_{\lambda z})dy=0$

$a_1x+b_1y=z$

reduce eq 1 to a separable eq., to variable x and z.

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Ex- $(x-2y+1)dx+(4x-3y-6)dy=0$

$\frac{4}{1} \neq \frac{+3}{+2}$

let $x=X+h$ and $y=Y+k$

$h-2k+1=0$

$4h-3k-6=0$

$4h-8k+4=0$

$4h-3k-6=0$

$-\quad+\quad+$

$-5k=-10$

$k=2,\ h=3$

eqn became $(X-2Y)dX+(4X-3Y)dY=0$

$\frac{dY}{dX}=\frac{2Y-X}{4X-3Y}$

let $Y=VX$

$\frac{dY}{dX}=V+X\frac{dv}{dX}$

$V+X\frac{dv}{dX}=\frac{2VX-X}{4X-3VX}$

$V+X\frac{dv}{dX}=\frac{2V-1}{4-3v}$

$X\frac{dv}{dX}=\frac{2V-1-4V+3V^2}{4-3v}$

$X\frac{dv}{dX}=\frac{3V^2-2V-1}{4-3V}$

$\frac{4-3V}{3V^2-2V-1}dV=\int\frac{dX}{X}$

Integrate both side

$\frac{2}{2}\int\frac{4-3V}{3V^2-2V-1}dv=\int\frac{dX}{X}$

$\frac{1}{2}\int\left[\frac{6V-2}{3V^2-2V-1}-\frac{6}{3V^2-2V-1}\right]dv=\int\frac{dX}{X}$

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$\frac{1}{2}\left[\frac{6v-2}{3v^2-2v-1}-\frac{6}{(v-1)(3v+1)}\right]=-\int\frac{dX}{X}$

$\frac{1}{2}\left[\frac{6v-2}{3v^2-2v-1}-\frac{3}{4(v-1)}+\frac{9}{4(3v+1)}\right]dv=-\frac{dX}{X}$

By integrate we get

$\frac{1}{2}\log|3v^2-2v-1|-\frac{3}{4}\log|v-1|+\frac{9}{4}\frac{\log|3v+1|}{3}=-\ln x+\ln c$

$\log|3v^2-2v-1|^2-3\log|v-1|+\log|3v+1|^3=\ln x^4+\ln c$

$\ln\left[\frac{x^4(3v^2-2v-1)^2(3v+1)^3}{(v-1)^3}\right]=\ln c$

Put $v=\frac{Y}{X}$ $\quad = X=x-3$

$Y=y-2$

Put the value

$(x+2y+3)dx+(2x+4y-1)dy=0$

$\frac{dm(x,y)}{dy}=2$ $\quad\frac{dN(x,y)}{dx}=2$

let $z=x+2y$

$dz=dx+2dy$

$dy=\frac{dz-dx}{2}$

eq became $(z+3)dx+(2z-1)\left(\frac{dz-dx}{2}\right)$

$\left[z+3-\frac{(2z-1)}{2}\right]dx+(2z-1)dz$

$\left[\frac{2z+6-2z+1}{2}\right]+(2z-1)dz$

$\frac{7dx}{2}+(2z-1)dz=0$

$(2z-1)dz=-\frac{7}{2}dx$

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Unit-2

Explicit method of solving higher-order linear diff. eqn.

A linear ~~diff~~ ordinary diff. eqn. of order ~~n~~ is the dependent variable y and the independent variable x is an eqn that is in, or can be expressed in the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=F(x)$$

where a_0 is not identically zero. We shall assume that $a_0, a_1, \dots a_n$ and F are continuous real fn. on a real interval $a \le x \le b$ and that $a_0(x) \neq 0$ for any x on $a \le x \le b$. The right hand number $F(x)$ is called non-homogenous term. If F is identically zero then equation is homogenous term.

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}(x)\frac{dy}{dx}=0$$

Ex 4.1 $\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+x^3y=e^x$

It is a linear ordinary diff. eqn of second order

Theorem 4.1:-

1. Consider the nth order linear differential eqn.

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=F(x)$$

where $a_0, a_1, \dots, a_n$ and F are continuous real fn on a real interval $a \le x \le b$ and $a_0(x) \neq 0$ for any x on $a \le x \le b$.

2. Let x_0 be any pt. on the interval $a \le x \le b$ and

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let $c_0, c_1, \ldots, c_{n-1}$ be n arbitary real constant

Conclusion:- There exist a unique sol. of f such that $f(x_0) = c_0, f'(x_0) = c_1, \ldots, f^{(n-1)}(x_0) = c_{n-1}$ and this solution is defined over the entire interval $a < x < b$.

Corollary:-

Hypothesis: Let f be a solution of n-th order homogenous linear diff. eqn.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

$f(x_0) = 0, f'(x_0) = 0, \ldots, f^{(n-1)}(x_0) = 0$

where x_0 is a pt. of the interval $a < x < b$ in which the coefficents are $a_0, a_1, \ldots, a_n$ are all continuous and $a_0(x) \neq 0$

Conclusion:- Then $f(x) = 0$ for all x on $a < x < b$

The Homogenous equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

Theorem:-

Hypothesis:- Let $f_1, f_2, \ldots, f_m$ be any m solution of homogenous linear diff. eqn.

Conclusion:- Then $c_1 f_1 + c_2 f_2 + \dots + c_m f_m$ is also a soln. of Homogenous equation, where $c_1, c_2, \dots, c_m$ are m arbitary constant

Defination:- If $f_1, f_2 \dots f_m$ are m given function and $c_1, c_2, \dots, c_m$ are m constant then the expression

$$c_1 f_1 + c_2 f_2 + \dots + c_m f_m$$

is called a linear combination of $f_1, f_2, \dots, f_m$

Theorem (Restated):- Any linear combination of solution of the homogenous diffrentation equation is also a soln. of homogenous eqn.

Ex:- Find the solution of

$$\frac{d^2y}{dx^2} + y = 0$$

if $f(x) = \sin x$ and $\cos x$

$\therefore c_1 f_1 + c_2 f_2 = c_1 \sin x + c_2 \cos x$

where c_1 and c_2 are any constant.

let ~~5~~ $c_1 = 5$, $c_2 = 2$

so eqn is $5 \sin x + 2 \cos x$

Defination:- The n function $f_1, f_2, \dots, f_n$ are called linearly dependent on $a < x < b$ if there exist constant $c_1, c_2, \dots, c_n$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$$

for all x such that $a \le x \le b$

Ex:- We.

Defination:- The n fun $f_1, f_2, \ldots, f_n$ are called linearly independent on the interval $a \le x \le b$ if they are not linearly dependent there. That is the function $f_1, f_2, \ldots, f_n$ are linearly independent on $a \le x \le b$ if the realtion

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

for all x such that $a \le x \le b$ implies that

$$c_1 = c_2 = \ldots = c_n = 0$$

In other word the linear combination $f_1, f_2, \ldots, f_n$ that is identically zero on $a \le x \le b$ is the trivial linear combination

$$0 \cdot f_1 + 0 \cdot f_2 + \ldots + 0 \cdot f_n$$

In particular two function f_1 and f_2 are linearly independent on $a \le x \le b$ if the relation

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x on $a \le x \le b$ implies that

$$c_1 = c_2 = 0$$

Theorem 4.3:-

The n-th order homogenous linear diff. eqn

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x) \frac{dy}{dx} + a_n(x) y =$$

always possesses n sol. that are linear

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independent. Further if $f_1, f_2, \ldots, f_n$ are n linearly independent sol^n of then every sol^n f can be expressed as a linear combination

$$c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$$

of these n linearly independent soln. by proper choice of constant $c_1, c_2, \ldots, c_n$.

Ex: **Defination**: If $f_1, f_2, \ldots, f_n$ are n linearly independent solution of n order homogenous linear diff. eqn.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

on $a \le x \le b$ then the set $f_1, f_2, \ldots, f_n$ is called a fundamental set of, and the function f defined by

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x), \quad a \le x \le b$$

where $c_1, c_2, \ldots, c_n$ are arbitary constant is called a general solution of on $a \le x \le b$

Ex: The sol. e^x, e^{-x} and e^{2x} of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0.$$

may be shown to be linearly independent for all. Thus e^x, e^{-x} and e^{2x} consitute a fundamental set and general soln. is

$$c_1 e^x + c_2 e^{-x} + c_3 e^{2x}.$$

where c_1, c_2 and c_3 are constant. We write this as

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}.$$

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Defination:-

Let $f_1, f_2, \ldots, f_n$ be n real fn. which has (n-1) derivative on a real interval. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

in which primes denotes derivatives is called the Wronskian of these n func. We must observe that $W(f_1, f_2, \ldots, f_n)$ is itself a real func. defined on $a \le x \le b$. Its value at x is denoted by $W(f_1, f_2, \ldots, f_n)(x)$ or by $W[f_1(x), f_2(x), \ldots, f_n(x)]$

Theorem: There n soln. $f_1, f_2, \ldots, f_n$ of the n-th order homogenous linear diff. eqn. are linearly independent on $a \le x \le b$ only if Wronskian of $f_1, f_2, \ldots, f_n$ is diff. from zero for some x on the interval $a \le x \le b$

Theorem: The Wronskian of n solution $f_1, f_2, \ldots, f_n$ of is either identically zero on $a \le x \le b$ or else is never zero on $a \le x \le b$.

Ex:- $\frac{d^2y}{dx^2} + y = 0$

Solution are $\sin x, \cos x$

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$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x = -1 \neq 0.$$

We concluded the solution are linearly independent

Ex:- The solution of e^x, e^{-x} and e^{2x} of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0.$$

are linearly independent on every interval for

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$\cancel{e^x} \cdot \cancel{e^{-x}} \cdot e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix}$$

$$e^{2x}\left[1(-4-2) - 1(4-2) + 1(1+1)\right]$$

$$= 6 - 2 + 2$$

$$-6e^{2x} \neq 0.$$

Reduction of order.

Hypothesis:- Let f be a non-trivial soln. of n-th order homogenous linear differential equation

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} = 0$$

Conclusion: The transformation $y = f(x)v$ reduce equation to an $(n-1)$st order homogenous equation linear diff. eqn. is dependent variable $w = dv/dx$.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

Let us make the transformation

$$y = f(x)v$$

$$\frac{dy}{dx} = f(x)\frac{dv}{dx} + f'(x)v$$

$$\frac{d^2y}{dx^2} = f(x)\frac{d^2v}{dx^2} + f'(x)\frac{dv}{dx} + f'(x)\frac{dv}{dx} + f''(x)v$$

$$\frac{d^2y}{dx^2} = f(x)\frac{d^2y}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v$$

$$a_0(x)\left[f(x)\frac{d^2y}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v\right] + a_1(x)\left[f(x)\frac{dv}{dx} + f'(x)v\right] + a_2(x)f(x)v = 0$$

$$a_0(x)f(x)\frac{d^2v}{dx^2} + \left[2a_0(x)f'(x) + a_1(x)f(x)\right]\frac{dv}{dx} + \left[a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)\right]v = 0$$

Since f is a solⁿ. the coefficient of v is zero, and the last equation reduces to

$$a_0(x)f(x)\frac{d^2v}{dx^2} + \left[2a_0(x)f'(x) + a_1(x)f(x)\right]\frac{dv}{dx} = 0$$

Letting $w = \frac{dv}{dx}$

$$a_0(x)f(x)\frac{dw}{dx} + \left[2a_0(x)f'(x) + a_1(x)f(x)\right]w = 0$$

$$\frac{dw}{w} = -\left[\frac{2f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}\right]dx,$$

Integrate both side

$$w = -\ln[f(x)]^2 - \int\frac{a_1(x)}{a_0(x)}dx + \ln c$$

$$w = \frac{c\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{[f(x)]^2}$$

$$v = \int \frac{c\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{[f(x)]^2}dx$$

$$y = f(x)\int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{[f(x)]^2}dx$$

$$w(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

$$= \begin{vmatrix} f(x) & g(x)v \\ f'(x) & f'(x)v + c \end{vmatrix}$$

$$= [f(x)]^2 v' = \exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right] + 0$$

Thus the linear combination

$$c_1 f + c_2 g$$

is the general solution

Theorem: Let f be a non-trivial solution of the second order homogenous linear diff. eqn.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

Conclusion I: The transformation $y = f(x)v$ reduces eqn. to the first order homogenous linear diff. eqn.

$$a_0(x) f(x)\frac{dw}{dx} + [2a_0(x) f'(x) + a_1(x) f(x)]w = 0$$

is the dependent variable $w = \frac{dv}{dx}$

Conclusion 2 - The particular soln.

$$w = \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)} dx\right]}{[f(x)]^2}$$

of eqn give rise to function v where

$$v(x) = \int \frac{\exp\left[\int \frac{a_1(x)}{a_0(x)} dx\right]}{[f(x)]^2} dx.$$

The function g defined by $g(x) = f(x)v(x)$ is then a soln. of the second order equation.

Conclusion 3 - The original know solution f and the "new" solution g linearly independent soln. and hence the general solution may be expressed as the linear combination

$$c_1 f + c_2 g$$

Ex. $y = x$ Given that $y = x$ is a sol. of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

find a linearly independent sol. by reducing the order

First observe that $y = x$ does not satisfy Then let

$$y = xv$$

Then $$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\Rightarrow \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

$$(x^2+1)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(x\frac{dv}{dx} + v\right) + 2vx = 0$$

$$x(x^2+1)\frac{d^2v}{dx^2} + 2(x^2+1)\frac{dv}{dx} - 2x^2\frac{dv}{dx} - \cancel{2xv} + \cancel{2vx} = 0$$

$$x(x^2+1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0$$

Let $w = dv/dx$

$$x(x^2+1)\frac{dw}{dx} + 2w = 0$$

$$\frac{dw}{dx} = \frac{-2w}{x(x^2+1)}$$

$$\frac{dw}{w} = \frac{-2dx}{x(x^2+1)}$$

$$\frac{dw}{w} = \left(\frac{-2}{x} + \frac{2x}{x^2+1}\right)dx$$

Int. both side

$$\log w = \log\left|\frac{1}{x^2}\right| + \log|(x^2+1)^2| + \log C$$

$$w = \frac{C(x^2+1)}{x^2}$$

Choosing $C=1$, $dv/dx = w$ to integrate to obtain a fn

$$v(x) = x - \frac{1}{x}$$

Now put $g = f(x)v$.

$$g(x) = \left(x - \frac{1}{x}\right)x$$

$$g(x) = x^2 - 1$$

<u>Non homogenous eqn:</u>

<u>Theorem:</u>

(non-homogenous)

<u>Hypothesis:</u> Let v be any sol. of the given n-th order linear diff. eqn. Let u be any solution of the corresponding homogenous equation.

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

Conclusion:- Then $u+v$ is also a sol. of the given (nonhomogeneous) equation.

Ex:- Observe that $y=x$ is a sol. of nonhomogeneous eq

$$\frac{d^2y}{dx^2}+y=x$$

and that $y=\sin x$ is a sol. of the corresponding homogenous equation.

Then by theorem the sum

$$\sin x+x$$

is also a sol. of the given non-homogenous equation.

Theorem:-

Hypothesis:-

Let y_p be a given solution of n-th order nonhomogeneous linear equation involving no arbitrary constant

$$y_c=c_1y_1+c_2y_2+\dots+c_ny_n$$

is the general sol. of the corresponding homogeneous eqn.

Conclusion:- Then every solution ϕ of the n-th order nonhomogenous equation can be expressed in form

$$y_c+y_p$$

that is $c_1y_1+c_2y_2+\dots+c_ny_n+y_p$

for suitable choice of the n arbitrary constants $c_1, c_2, \dots, c_n$

Defination:- Consider the n-th order (nonhomogenous) linear diff. eqn.

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=F(x)$$

and the corresponding homogenous equation.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

★ The general sol. is called the complementary of func. of equation. We shall denote this by y_c.

★ Any particular sol. of involving no arbitary const. is called a particular integrble. We shall denote this by y_p.

★ The sol. $y_c + y_p$ where y_c is the complementary func. and y_p is particular integral is called the general sol.

Ex - Consider the equation

$$\frac{d^2y}{dx^2} + y = x.$$

The complentary function is general sol.

$$y_c = c_1 \sin x + c_2 \cos x.$$

of the corresponding homogenous eq.

$$\frac{d^2y}{dx^2} + y = 0$$

A particular integeral is given by

$$y_p = x$$

Thus the general sol. of given eqn may be written

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + x.$$

Theorem:-
Hypothesis
Let f_1 be a particular integral of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F_1(x)$$

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2. Let f_2 be a particular integral of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F_2$$

Conclusion: Then, $K_1 f_1 + K_2 f_2$ is a particular integral of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = K_1F_1 + K_2F_2$$

Ex – Suppose we seek a particular integral of

$$\frac{d^2y}{dx^2} + y = 3x + 5\tan x$$

We may consider the two eqns →

$$\frac{d^2y}{dx^2} + y = x$$

and $$\frac{d^2y}{dx^2} + y = \tan x.$$

We have noted that

$$y = x.$$

Further we can that a particular eqn is given by

$$y = -(\cos x)\log|\sec x + \tan x|.$$

Therefore, applying theorem

$$y = 3x - (\cos x)\log|\sec x + \tan x|$$

This ex. makes the utility of theorem. The particular integral $y = x$ can be quickly determined by the method (or by direct inspection) whereas the particular integral

$$y = -(\cos x)\ln|\sec x + \tan x|.$$

The homogenous linear eqn. with constant coefficient

Distinct Real root

Suppose the roots of are the n distinct real number $m_1, m_2, \ldots, m_n$

Then $e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}$.

Theorem: Consider the n-th order homogenous linear diff. eqn. with constant coefficient. If the auxilary eqn has the n distinct real root $m_1, m_2, \ldots, m_n$ then the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}.$$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Auxiliary eqn. is

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$m = 2, m = 1.$$

The roots are real and distinct.

Thus e^x and e^{2x} are sol. and the general sol. may be written

$$y = c_1 e^x + c_2 e^{2x}$$

We verify that the e^x and e^{2x} are indeed linearly independent. Their Wronskian is

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

Case 2: Repeated real roots

Ex:-

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Auxilary eqn

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

The solution may be written as

$$y = C_1 e^{3x} + C_2 x e^{x}$$

The solution will be

$$y = C_1 e^{3x} + C_2 \cancel{x} e^{x}$$

∴ We must a find linear Independent solution because this have same root

So, $y = e^{3x} v$

where v is to determined

$$\frac{dy}{dx} = e^{3x}\frac{dv}{dx} + 3e^{3x}v$$

$$\frac{d^2y}{dx^2} = e^{3x}\frac{d^2v}{dx} + 3e^{3x}\frac{dv}{dx} + 9e^{3x}v +$$

Subsitute in main equation

$$e^{3x}\frac{d^2v}{dx^2} + 6e^{3x}\frac{dv}{dx} + 9e^{3x}v - 6e^{3x}\frac{dv}{dx} - 18e^{3x}v + 9e^{3x}v = 0$$

$$e^{3x}\frac{d^2v}{dx^2} = 0$$

Let $w = dv/dx$

$$e^{3x}\frac{dw}{dx} = 0$$

$$\Rightarrow \frac{dw}{dx} = 0$$

Integrate
$$w = c$$
We choose arbitrary constant $c = 1$ and $dv/dx = w$.
$$\frac{dv}{dx} = w = c$$
Integrate both side
$$\int dv = \int dx$$
$$v(x) = x + c_0$$
where c_0 is an arbitrary constant
Multiply both side by e^x
$$v(x) e^{3x} = (x + c_0) e^x$$
let $c_0 = 0 \Rightarrow v(x) e^{3x} = x e^x$
and corresponding double root 3 we find the solution
$$e^{3x} \text{ and } x e^{3x}$$
Thus general sol is $c_1 e^{3x} + c_2 x e^{3x}$
suppose we have double root m, we should expect that e^{mx}, and $x e^{mx}$. This is suppose the root are double real root m and the $(n-2)$ distinct real root
$$m, m_2, \ldots, m_n \}$$
The linearly independent sol are
$$e^{mx}, x e^{mx}, e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_{n-2} x}$$
and the general sol may be
$$y = c_1 e^{mx} + x c_2 e^{mx} + c_3 e^{m_1 x} + c_4 e^{m_2 x} + \ldots + c_n e^{m_{n-2} x}$$

If ~~ass~~ any equation are triple root then sol will
$$e^{mx}, \ x e^{mx} \text{ and } x^2 e^{mx}$$
The corresponding part of general sol may be
$$(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$$

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Theorem:-

1 Consider the n-th order homogenous linear diff. eqn with constant cofficient. If the auxilary eqn has the real root m occuring k times, then general soln corresponding to this k-fold repeated root
$$(c_1 + c_2x + c_3x^2 + c_k x^{k-1}) e^{mx}$$

2 If further the remaining root of auxiliary eqn are the distinct number $m_{k+1}, \ldots, m_n$ then general sol. is
$$y = c_1 + c_2x + c_3x^2 + \ldots + c_k x^{k-1}) e^{mx} + c_{k+1} e^{m_{k+1}x} + \ldots + c_n e^{m_n x}$$

3 If however any of the remaining root are also repeated then part of general sol. of corresponding to each of these other repeated roots are expression similar to that corresponding to m in part 1

Ex:- $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$

Auxiliary eqn
$$m^3 - 4m^2 - 3m + 18 = 0$$
has root 3, 3, -2. The gen. sol. is
$$y = c_1e^{3x} + xc_2e^{3x} + c_3e^{-2x}$$

Case 3:- Conjugate Complex root

Now suppose that auxilary equation has complex number as a non repeated root. Then since the cofficient are real the conjugate complex no is a-bi is a also a non repeated root. The general eq. is

$K_1 e^{(a+bi)x} + K_2 e^{(a-bi)x}$,

where K_1 and K_2 are arbitrary constant. The solution defined by $e^{(a+bi)x}$ and $e^{(a-bi)x}$ are complex fn. of real variable x. It is desirable two linearly independent sol. This can be accomplished by Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which hold for all real θ. Using this we have

$$K_1 e^{(a+bi)x} + K_2 e^{(a-bi)x} = \cancel{K_1 e^{x}\, e^{bi}}$$

$$= K_1 e^{ax} e^{bix} + K_2 e^{ax} e^{-bix}$$

$$e^{ax}\left[K_1(\cos bx + i\sin bx) + K_2(\cos bx - i\sin bx)\right]$$

$$e^{ax}\left[(K_1+K_2)\cos bx + i\sin(K_1-K_2)\right]$$

$$e^{ax}(c_1\cos bx + c_2\sin bx)$$

$$\therefore\ K_1 + K_2 = c_1 \text{ and } K_1 - K_2 = c_2$$

Theorem:- Consider the n-th order homogenous diff. eqn. with constant cofficient. If the auxiliary eqn. has the conjugate complex root $a+bi$ and $a-bi$ neither repeated, then the corresponding part of general sol. may be written

$$y = e^{ax}(c_1\sin bx + c_2\cos bx)$$

2 If however $a+bi$, $a-bi$ are each k-fold root of auxilary eqn then the corresponding part of general soln. may be written

$$y = e^{ax}\left[(c_1 + c_2x + c_3x^2 + \dots + c_k x^{k-1})\sin bx + (c_{k+1} + c_{k+2}x + c_{k+3}x^2 + \dots + c_{2k}x^{k-1})\cos bx\right]$$

$$y = c_1\sin x + c_2\cos x$$

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☆ The method of undetermined coefficient

We now consider the (nonhomogenous) diff. eqn

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = F(x)$$

where the coefficient $a_0, a_1, \ldots, a_n$ are constant but where the nonhomogenous term F is a nonconstant function of x. Recall the gen. sol. may be written

$$y = y_c + y_p$$

∴ y_c is complementary function
∴ y_p is particular integral

Ex: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x}$

Auxiliary eqn

$$m^2 - 2m - 3 = 0$$
$$m^2 - 3m + m - 3 = 0$$
$$m(m-3) + 1(m-3) = 0$$
$$m = 3,\ m = -1$$

linear independent function e^{3x}, e^{-x}

Thus we assume a particular sol. of form

$$y_p = Ae^{4x}$$
$$y'_p = 4Ae^{4x}$$
$$y''_p = 16Ae^{4x}$$

Substitute in eqn

$$16Ae^{4x} - 8Ae^{4x} - 3Ae^{4x} = 2e^{4x}$$
$$5Ae^{4x} = 2e^{4x}$$
$$5A = 2$$
$$A = 2/5$$

Substitute in $y_p = \frac{2}{5}e^{4x}$

General eqn is $y = c_1e^{3x} + c_2e^{-x} + \frac{2}{5}e^{4x}$

∴ We take that function in y_p which is on RHS so after diff. we can equate them

Defination :- We shall call a function a UC function if it is either a func. defined by one of the following:

- x^n, where n is a positive integer or zero.
- e^{ax}, where a is constant $\neq 0$
- $\sin(bx+c)$ where b and c are constant, $b \neq 0$

UC function (Undetermined coefficient)

x^n	e^{x}	$\sin(ax+b)$	$\cos(ax+b)$
n is +ve integer or Zero	a constant which is non zero	$a \neq 0$, a, b const	$a \neq 0$, a, b const
$x^n, nx^{n-1}, n(n-1)x^{n-2}, \ldots n!$	$e^{ax}, ae^{ax}, a^2e^{ax}$	$\sin(ax+b)$, $a\cos(ax+b)$, $-a^2\sin(ax+b)$	$\cos(ax+b)$, $-a\sin(ax+b)$, $-a^2\cos(ax+b)$
$\{x^n, x^{n-1}, x^{n-2}, \ldots x, 1\}$	$\{e^{ax}\}$	$\{\sin(ax+b), \cos(ax+b)\}$	$\{\cos(ax+b), \sin(ax+b)\}$

UC set :- Consider a UC func. f. The set of func. itself and all linearly independent UC func which the successive derivative of f are either constant multiply or linear comb. is called UC set of f.

Ex. $f(x) = x^3$

$x^3 \quad 3x^2 \quad 6x \quad 1$

$x^3 \quad x^2 \quad x \quad 1$

UC set of $x^3 = \{x^3, x^2, x, 1\}$

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2. $f(x) = e^{-2x}$ $e^{-2x} \downarrow e^{-2x}$ $-2e^{-2x} \downarrow e^{-2x}$

UC set of $e^{-2x} = \{e^{-2x}\}$.

3. $f(x) = x^2 \sin x$

$x^2 \rightarrow x^2 \downarrow x^2$ $2x \downarrow x$ $\rightarrow 2 \downarrow 1$

$\sin x \rightarrow \{\sin x \quad \cos x\}$

UC set of $x^2 \sin x = \{x^2 \sin x, x^2 \cos x, x \sin x, x \cos x, 2\sin x, 2\cos x\}$.

UC function	UC set
x^n	$\{x^n, x^{n-1}, \ldots, x, 1\}$.
e^{ax}	$\{e^{ax}\}$.
$\sin(ax+b)$	$\{\sin(ax+b), \cos(ax+b)\}$
$x^n \cos(ax+b)$	$\{x^n \cos(ax+b), x^{n-1}\cos(ax+b), \ldots, x\cos(ax+b), \cos(ax+b)\}$
$x^n \sin(ax+b)$	$\{x^n \sin(ax+b), x^{n-1}\sin(ax+b), \ldots, x\sin(ax+b), \sin(ax+b)\}$
$e^{ax}\sin(ax+b)$	$\{e^{ax}\sin(ax+b), e^{ax}\cos(ax+b)\}$
$e^{ax}\cos(ax+b)$	$\{e^{ax}\cos(ax+b), e^{ax}\sin(ax+b)\}$
$x^n e^{ax}\sin(ax+b)$	$\{x^n e^{ax}\sin(ax+b), x^n e^{ax}\cos(ax+b), x^{n-1}e^{ax}\sin(ax+b), x^{n-1}e^{ax}\cos(ax+b), \ldots, xe^{ax}\sin(ax+b), xe^{ax}\cos(ax+b), e^{ax}\sin(ax+b), e^{ax}\cos(ax+b)\}$.

Eg. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2e^x$

Auxiliary eqn.

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$m = 1, 2$

$y_c = C_1e^x + C_2e^{2x}$ ∴ C_1 & C_2 are const.

y_p $x^2e^x = \alpha\, x^2e^x, xe^x, e^x$

$y_p = Ax^3e^x + Bx^2e^x + Ce^x$

$y_p' = 2Axe^x + Ax^2e^x + Be^x + Bxe^x + Ce^x$

$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x + Be^x + Bxe^x + Ce^x$

$= Ax^2e^x + 4Axe^x + 2Ae^x + Be^x + Bxe^x + Ce^x$

$Ax^2e^x + 4Axe^x + 2Ae^x + Be^x + Bxe^x + Ce^x - 6Axe^x - 3Ax^2e^x - 3Be^x - 3Bxe^x - 3Ce^x + 2Ax^2e^x + 2Bxe^x + 2Ce^x$

$6Axe^x - 3Ax^3e^x + 2Bxe^x - 2Bxe^x = x^2e^x = x^2e^x$

On comparing

$-3A = 1$ $\quad$ $6A - 2B = 0$ $\quad$ $2B - C = 0$

$A = -\frac{1}{3}$ $\quad$ $6A = 2B$ $\quad$ $2B = C$

$B = 3A$ $\quad$ $C = -2$

$B = -1$

$y_p = -\frac{1}{3}x^3e^x - x^2e^x - 2xe^x$

$y = y_c + y_p$

$= C_1e^x + C_2e^{2x} - \frac{1}{3}x^3e^x - x^2e^x - 2xe^x$

Q. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x$

Auxelary eqn.

$m^2 - 2m - 3 = 0$

$m(m-3) + 1(m-3) = 0$

$m = -1, 3$

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$y_C = C_1e^{-x} + C_2e^{3x}$

$2e^x - 10\sin x = \{e^x, \sin x, \cos x\}$

$y_p = Ae^x + B\sin x + C\cos x$

$y_p' = Ae^x + B\cos x - C\sin x$

$y_p'' = Ae^x - B\sin x - C\cos x$

$Ae^x - B\sin x - C\cos x - 2Ae^x - 2B\cos x - 2C\sin x - 3Ae^x$
$- 3B\sin x - 3C\cos x$

$-4Ae^x + (-4B + 2C)\sin x + \cos x(-4C - 2B) = 2e^x - 10\sin x$

On comparing

$-4A = 2 \qquad -4B + 2C = -10 \qquad -4C - 2B = 0.$

$A = -\frac{1}{2} \qquad -2B + C = -5$

$C = -5 + 2B$

$-4C - 2B = 0.$

$-4(-5 + 2B) - 2B = 0$

$20 - 8B - 2B = 0$

$10B = 20$

$B = 2.$

$C = -5 + 2B$

$C = -5 + 4$

$C = -1.$

$y_p = -\frac{1}{2}e^x + 2\sin x - \cos x$

$y = C_1e^{3x} + C_2e^{-x} - \frac{e^x}{2} + 2\sin x - \cos x.$

Methods of variation of parameter

$\frac{d^2y}{dx^2} + y = \tan x$

Consider the general second order diff. eqn

$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x) \quad (1)$

let y_1 and y_2 be two linear independent soln of the corresponding homogenous diff. eqn.

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Let $y_p = v_1(x) y_1(x) + v_2(x) y_2(x)$ be a particular integral.
Find $v_1(x)$ and $v_2(x)$.

Then $y'_p = v_1(x) y'_1(x) + v'_1(x) y_1(x) + v_2(x) y'_2(x) + v'_2(x) + y_2(x)$.

Now putting $v'_1(x) y_1(x) + v'_2(x) y_2(x) = 0$.

$\Rightarrow y'_p = v_1(x) y'_1(x) + v_2(x) y'_2(x)$.

$\Rightarrow y''_p = v_1(x) y''_1(x) + v'_1(x) y'_1(x) + v_2(x) y''_2(x) + v'_2(x) y'_2(x)$

Substitute the value in eq. (1) we get

$a_0[v_1(x) y''_1(x) + v'_1(x) y'_1(x) + v_2(x) y''_2(x) + v'_2(x) y'_2(x)] + a_1[v_1(x) y'_1(x) + v_2(x) y'_2(x)] + a_2[v_1(x) y_1(x) + v_2(x) y_2(x)] = f(x)$.

$\Rightarrow a_0 v_1(x) y''_1(x) + a_0 v'_1(x) y'_1(x) + a_0 v_2(x) y''_2(x) + a_0 v'_2(x) y'_2(x) + a_1 v_1(x) y'_1(x) + a_1 v_2(x) y'_2(x) + a_2 v_1(x) y_1(x) + a_2 v_2(x) y_2(x) = f(x)$.

$\Rightarrow v_1(x)[a_0 y''_1(x) + a_1 y'_1(x) + a_2 y_1(x)] + v_2(x)[a_0 y''_2 + a_1 y'_2(x) + a_2 y_2(x)] + a_0[v'_1(x) y'_1(x) + v'_2(x) y'_2(x)] =$

Since y_1 and y_2 are the sol. of homogenous diff. eqn. corresponding to (1)

$\therefore a_0(v'_1(x) y'_1(x) + v'_2(x) y'_2(x)) = f(x)$

$y'_1(x) v'_1(x) + y'_2(x) v'_2(x) = \frac{f(x)}{a_0}$

So, $y_1(x) v'_1(x) + y_2(x) v'_2(x) = 0$

$y'_1(x) v'_1(x) + y'_2(x) v'_2(x) = \frac{f(x)}{a_0}$.

Common Rule:

$$a_1(x)+b_1(y)+c_1(x)=d_1$$
$$a_2(x)+b_2(y)+c_2(x)=d_2$$
$$a_3(x)+b_3(y)+c_3(x)=d_3$$

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$v_1'(x)=-\frac{y_2(x)\,f(x)}{a_0 W(y_1(x),y_2(x))} \qquad v_2'(x)=\frac{y_1(x)\,f(x)}{a_0 W(y_1(x),y_2(x))}$$

$$v_1(x)=-\int_x \frac{y_2(t)\,f(t)}{a_0 W(y_1(t),y_2(t))} \qquad v_2(x)=\int_x \frac{y_1(t)\,f(t)}{a_0 W(y_1(t),y_2(t))}$$

$$y_p=v_1(x)y_1(x)+V_2(x)\,y_2(x).$$

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Eg $\frac{d^2y}{dx^2} + y = \tan x$

The corresponding diff. eqn is:

$\frac{d^2y}{dx^2} + y = 0$

Auxiliary eqn:

$m^2 + 1 = 0$

$(m+i)(m-i) = 0$

$m = \pm i$

$y_c \Rightarrow c_1 \sin x + c_2 \cos x$ ∴ where c_1 and c_2 are a

Assume that a particular soln. of eq.(1) is

$y_p = v_1(x) \sin x + v_2(x) \cos x$

$y_1 = \sin x$, $y_2 = \cos x$.

Now

$$W(y_1(x), y_2(x)) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 = -1$$

$$v_1(x) = -\int \frac{y_2(t) f(t)}{W(y_1(t) y_2(t))} dt$$

$$= -\int \frac{\cos t \tan t}{-1} dt$$

$$= \int \frac{\cancel{\cos t} \sin t}{\cancel{\cos t}} dt = \int \sin t \, dt$$

$v_1(x) = -\cos x + c_3$ ∴ c_3 is a constant of Int

$$v_2(x) = \int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt$$

$$= \int \frac{\sin t \tan t}{-1} dt$$

$$-\int \frac{\sin^2 t}{\cos t} dt = -\int \frac{(1-\cos^2 t)}{\cos t} dt$$

$$= -\int (\sec t - \cos t) dt =$$

$$= -[\log|\sec x + \tan x| - \sin x] + c_4$$

$$= \sin x - \log|\sec x + \tan x| + c_4$$

$\therefore c_4$ is const.

$$y_p = v_1(x) y_1(x) + v_2(x) y_2(x)$$

$$= (-\cos x + c_3) \sin x + \cos x (\sin x - \log|\sec x + \tan x| + c_4)$$

$$= -\sin x \cos x + c_3 \sin x + \sin x \cos x - \cos x \log|\sec x + \tan x| + c_4 \cos x$$

$$= -\cos x \log|\sec x + \tan x| + c_3 \sin x + c_4 \cos x.$$

$$y = y_c + y_p$$

$$= c_1 \sin x + c_2 \cos x - \cos x \log|\sec x + \tan x| + c_3 \sin x + c_4 \cos x$$

$$= \sin x (c_1 + c_3) + \cos x (c_2 + c_4) - \cos x \log|\sec x + \tan x|$$

Cauchy euler equation

Cauchy euler equation is a diff. eqn. of the form.

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = f(x)$$

where $a_0, a_1, \dots, a_n$ are constant

Then $x = e^t$ reduces this diff. eqn. to a linear diff. eqn. with constant co-efficient

Consider the second order cauchy-euler diff. eqn.

$$a_0 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = f(x) \quad -(1)$$

Let $x = e^t$

$\log x = t$

$$\frac{1}{x} = \frac{dt}{dx}$$

Consider $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2} \times \frac{dt}{dx}$$

$$-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

Substitute this value in eq 1 we get

$$a_0x^2\left(\frac{d^2y}{dt^2}\frac{1}{x^2} - \frac{1}{x^2}\frac{dy}{dt}\right) + a_1x\left(\frac{1}{x}\frac{dy}{dt}\right) + a_2y = f(e^t)$$

$$a_0\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1\left(\frac{dy}{dt}\right) + a_2y = f(e^t)$$

$$a_0\frac{d^2y}{dt^2} + (a_1 - a_0)\left(\frac{dy}{dt}\right) + a_2y = f(e^t)$$ → second order linear diff eqn with constant coefficient

Ex. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$

Let $x = e^t$

$\log x = t$

$$\frac{1}{x} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = e^{3t}$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

Auxiliary eqn:

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$m = 1, 2.$

$y_c(t) = c_1 e^{t} + c_2 e^{2t}$; c_1 & c_2 are arbitrary const.

RHS of eqn 2 is e^{3t}

$\therefore$ UC set is $\{e^{3t}\}$.

Particular soln of eq 2 is of the form

$y_p = Ae^{3t}$ (where A is a const. to be determined)

$$\frac{dy_p}{dt} = 3Ae^{3t}$$

$$\frac{d^2y_p}{dt^2} = 9Ae^{3t}$$

$$9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} = e^{3t}$$

$$2Ae^{3t} = e^{3t}$$

$$A = 1/2$$

$$\therefore y_p = \frac{e^{3t}}{2}$$

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$$y(t) = y_c(t) + y_p(t)$$
$$= c_1 e^{t} + c_2 e^{2t} + \frac{e^{3t}}{2}$$

Put $x = e^{3t}$

$$= c_1 x + c_2 x^2 + \frac{x^3}{2}.$$

2. $x^3 \frac{d^3y}{dx^2} - 4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} - 8y = 4 \log x$

~~Put~~ let $x = e^t$

$$\log x = t$$

$$\frac{1}{x} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$$

$$x \frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d^2y}{dt^2} \times \frac{dt}{dx}$$

$$x^2 \frac{d^2y}{dx^2} = -\frac{dy}{dt} + \frac{d^2y}{dt^2}$$

$$\frac{d^3y}{dx^3} = \frac{2}{x^3} \frac{dy}{dt} - \frac{1}{x^3} \frac{d^2y}{dt^2} - \frac{2}{x^3} \frac{d^2y}{dt^2} + \frac{1}{x^3} \frac{d^3y}{dt^3}$$

$$\frac{d^3y}{dx^3} = \frac{2}{x^3} \frac{dy}{dt} - \frac{3}{x^3} \frac{d^2y}{dt^2} + \frac{1}{x^3} \frac{d^3y}{dt^3}$$

$$x^3 \frac{d^3y}{dt^3} = 2\frac{dy}{dt} - 3\frac{d^2y}{dt^2} + \frac{d^3y}{dt^3}$$

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8\frac{dy}{dt} - 8y = 4\log$$

$$\frac{d^3y}{dt^3} - 7\frac{d^2y}{dt^2} + 14y - 8y = 4\log x$$

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Auxilary eqn.

$m^3-7m^2+14m-8=0$.

Put $m=1$.

$(1)^3-7(1)^2+14(1)-8 = 1-7+14-8=0$

So ~~m=~~ $m^3 \quad -7m^2 \quad +14m \quad -8$

$1 \mid \quad 1 \quad -7 \quad 14 \quad -8$

$(m-1)\overline{)m^3-7m^2+14m+8}(m^2-6m+8$

m^3-m^2

$- \quad +$

$-6m^2+14m$

$-6m^2+6m$

$+ \quad -$

$8m+8$

$8m+8$

$- \quad -$

$\times$

$m^2-6m+8 = m^2-4m-2m+8$

$= m(m-4)-2(m-4)$

$= m = 4, 2$

So roots are $1, 2, 4$.

$y_c(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{4t}$.

UC set of $4\log x = \{t, 1\}$.

$y_p = At+B$

$y'_p = A$

$y''_p = 0 \quad , \quad y'''_p = 0$

So $14A - 8At - 8B = 4t$

$-8At = 4t$

$A = -1/2$

$14A = 8B \qquad B = -7/8$.

$y_p = -\frac{t}{2} - \frac{7}{8}$

$y(t) = y_c(t) + y_p(t)$

$y(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{4t} - \frac{t}{2} - \frac{7}{8}.$

$y(x)$ Put $e^t = x$

$y(x) = c_1 x + c_2 x^2 + c_3 x^4 - \frac{\log x}{2} - \frac{7}{8}$

☆ $x\frac{dy}{dx} = \frac{dy}{dt} = r$ $\quad \boxed{\because \frac{dy}{dt} = r}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} = r(r-1)$

$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = r(r-1)(r-2)$

$x^4\frac{d^4y}{dx^4} = r(r-1)(r-2)(r-3)$

$= (r^2 - r)(r^2 - 5r + 6)$

$= r^4 - 5r^3 + 6r^2 - r^3 + 5r^2 + 6r.$

$= r^4 - 6r^3 + 11r^2 + 6r$

$\frac{d^4y}{dt^4} - 6\frac{d^3y}{dt^3} + 11\frac{d^2y}{dt^2} + 6\frac{dy}{dt}$

$x^n\frac{d^ny}{dx^n} = r(r-1)(r-2) \ldots (r-(n-1))$

Some theorem on the second order homogenous linear diff. eqn

Theorem: Consider a second order homogenous linear diff. eqn.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0.$$

where a_0, a_1 and a_2 are continuous real value func. On a real interval $a \le x \le b$ and $a_0(x) \ne 0$ for any $x \in [a, b]$. let x_0 be any pt. of the interval $[a, b]$ and c_0 and c_1 be two constant.

Then there exist a unique soln. f of (1) such that $f(x_0) = c_0$ and $f'(x_0) = c_1$.
and this sol. is defined over the entire interval $a \le x \le b$.
In particular the unique f on (1) is such that $f(x_0) = 0$ and $f'(x)_0 = c$
is the function f such that $f(x) = 0 \ \forall x \in [a, b]$.

Theorem B. Two homogenous linear algebraic equation in two unknown have non-trivial soln. if and only if the determinant of the coefficient of the system is equal to zero.

$$a_1x + b_1y = 0$$
$$a_2x + b_2y = 0$$

$$2x + y = 0$$
$$x - y = 0$$
$$x = 0, y = 0$$
consistent (unique)

$Ax = b$
$x = A^{-1}B$
$\det A \ne 0$

consistent
$$2x + 2y = 0$$
$$x + y = 0, \quad y = -x$$

Infinitely many solution
$x = 0, y = 0$
$x = 1, y = -1$
$x = 2, y = -2$

det $= 0$ Non trivial soln.
det $\ne 0$ Trivial solution.

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Theorem 1:- Two linear algebraic eqn. in two unknowns have unique soln. if the determinant of the co-efficient of this system is not equal to zero.

Theorem: Let f_1 and f_2 be two solution of homogeneous linear diff. eqn:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad -(1)$$

On $a \leq x \leq b$ and let c_1 and c_2 be any two arbitrary constant. Then the linear combination $c_1f_1 + c_2f_2$ of the function f_1 and f_2 is also a solution of (1) on $a \leq x \leq b$

Proof: Since f_1 and f_2 are solution of 1

$$a_0(x)\frac{d^2f_1}{dx^2} + a_1(x)\frac{df_1}{dx} + a_2(x)f_1(x) = 0 \quad -(2)$$

and $$a_0(x)\frac{d^2f_2}{dx^2} + a_1(x)\frac{df_2}{dx} + a_2(x)f_2(x) = 0 \quad -(3)$$

let $f = c_1f_1 + c_2f_2$

f is solution of (1)

$$a_0(x)\frac{d^2f}{dx^2} + a_1(x)\frac{df}{dx} + a_2(x)f(x) = a_0(x)\left[c_1\frac{d^2f_1}{dx^2} + c_2\frac{d^2f_2}{dx^2}\right]$$

$$+ a_1(x)\left[c_1\frac{df_1}{dx} + c_2\frac{df_2}{dx}\right] + a_2(x)\left[c_1f_1(x) + c_2f_2\right]$$

$$= c_1\left[a_0(x)\frac{d^2f_1}{dx^2} + a_1(x)\frac{df_1}{dx} + a_2(x)f_1\right] +$$

$$c_2\left[a_0(x)\frac{d^2f_2}{dx^2} + a_1(x)\frac{df_2}{dx} + a_2(x)f_2\right]$$

Using eq (2) and (3) we get.

$$a_0(x)\frac{d^2 f}{dx^2} + a_1(x)\frac{df}{dx} + a_2(x)f(x) = 0.$$

$\Rightarrow f$ is solution of 1

Theorem 2:- Consider the second order homogenous linear diff. eqn

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad -(4)$$

where a_0, a_1 and a_2 are continous function on $a \leq x \leq b$ and $a_0(x) \neq 0 \ \forall x \in [a,b]$.
Then there exist a set of two linearly independent solution of 4 on $a \leq x \leq b$.

Proof: Let x_0 be any point of interval $[a,b]$. Then by theorem ⓐ there exists a unique sol. of f_1 of (4) such that

$f_1(x_0) = 1$ and $f_1'(x_0) = 0$

and unique sol. f_2 such that

$f_2(x_0) = 0$ and $f_2'(x_0) = 1$.

y_1, y_2
$\Rightarrow c_1 y_1 + c_2 y_2 = 0$
$\Rightarrow c_1 = 0 \ \& \ c_2 = 0$

To show: f_1 and f_2 are linearly independent. On the contrary, assume that f_1 and f_2 are not linearly independent.

$\Rightarrow f_1$ and f_2 are dependent.

$\Rightarrow$ there exists constants c_1 and c_2 not both zero such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \qquad \forall x \in [a,b]$$
$$c_1 f_1'(x) + c_2 f_2'(x) = 0 \qquad \forall x \in [a,b]$$

In particular for $x = x_0$ we have.

$$c_1 f_1(x_0) + c_2 f_2(x_0) = 0$$
$$c_1 f_1'(x_0) + c_2 f_2'(x_0) = 0$$

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix}$$

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$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\Rightarrow c_1 = 0$ and $c_2 = 0$
Our assumption is wrong. Hence f_1 and f_2 are linearly independent.

Theorem 3:- Two soln. f_1 and f_2 of the second order homog linear diff. eqn.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad -(A)$$

are linearly independent on $a \leq x < b$ if and only if the value of the Wronskian of f_1 and f_2 is equal to zero for all $x \in [a, b]$.

$$W(f_1(x), f_2(x)) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = 0 \text{ for all } x \in [a, b]$$

Proof:- let f_1 and f_2 be linearly independent on $a \leq x \leq$ then there exists constant c_1 and c_2 are not both zero such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \text{ for all } x \in [a, b]$$

To show $W(f_1(x), f_2(x)) = 0$ for all $x \in [a, b]$.
As $c_1 f_1(x) + c_2 f_2(x) = 0 \ \forall x \in [a, b]$.
$c_1 f_1'(x) + c_2 f_2'(x) = 0 \ \forall x \in [a, b]$.

Let x_0 be any arbitrary constant point of $[a, b]$
Then

$$c_1 f_1(x_0) + c_2 f_2(x_0) = 0$$
$$c_1 f_1'(x_0) + c_2 f_2'(x_0) = 0$$

Here both c_1 and c_2 not zero

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$$\Rightarrow \begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix} = 0.$$

Since x_0 is an arbitrary pt. of $[a,b]$.

$$\Rightarrow \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = 0 \text{ for all } x \in [a,b].$$

$W(f_1(x), f_2(x)) = 0$ for all $x \in [a,b]$.

Conversely assume that $W(f_1(x), f_2(x)) = 0$ for all $x \in [a,b]$.

i.e. $\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = 0$ for all $x \in [a,b]$.

Let x_0 be an arbitrary point of $[a,b]$. Then

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix} = 0$$

Then $\left.\begin{array}{l} c_1 f_1(x_0) + c_2 f_2(x_0) = 0 \\ c_1 f_1'(x_0) + c_2 f_2'(x_0) = 0 \end{array}\right] \quad -(1).$

Then the system of eq (1) has a non trivial soln.

Consider the function $f(x) = c_1 f_1(x) + c_2 f_2(x)$

Then $f(x)$ is solution of (1)

$f(x_0) = c_1 f_1(x_0) + c_2 f_2(x_0) = 0$ By (1)

$f'(x_0) = c_1 f_1'(x_0) + c_2 f_2(x_0) = 0$ By (1)

$\Rightarrow f(x) = 0$ for all $x \in [a,b]$

$c_1 f_1(x_0) + c_2 f_2(x_0) = 0$ for all $x \in [a,b]$.

Hence c_1 and c_2 are not both zero

c_1 & Hence f_1 and f_2 are linearly dependent.

f_1 and f_2 are dependent $\Rightarrow W(f_1, f_2) = 0$ for all $x \in [a, b]$

$P \Rightarrow Q$ Natural no $\Rightarrow$ whole no

$-Q \Rightarrow -P$ Not whole $\Rightarrow$ Not natural

f_1 and f_2 are dependent $\Rightarrow W(f_1(x), f_2(x)) = 0 \ \forall x \in [a, b]$

$W(f_1(x), f_2(x)) \neq 0$ for some $x \in [a, b]$ $\Rightarrow$ f_1 and f_2 are independent

$W[f_1(x), f_2(x)] = 0 \ \forall x \in [a, b] \Rightarrow f_1$ and f_2 are dependent.

f_1 and f_2 are independent $\Rightarrow W(f_1(x) f_2(x)) \neq 0$ for some $x \in [a, b]$

f_1 and f_2 are independent $\nRightarrow W(f_1(x), f_2(x)) \neq 0$ for some $x \in [a, b]$

Theorem 4: Two function f_1 and f_2 of the second order homogenous linear diff eqn

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0$$

are linearly independent on $a \leq x \leq b$ if and only if the value of Wronskian of f_1 and f_2 is non zero for some $x \in [a, b]$.

Theorem 1: The value of wronskian of two function f_1, f_2 of diff. eqn

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad -(1)$$

Either is zero for all x on $a \leq x \leq b$ or is zero for no x on $a \leq x \leq b$

Proof:- Case I:- If f_1 and f_2 are linearly dependent on $a \le x \le b$. Then Wronskian of f_1 and f_2 is equal to zero for all x on $a \le x \le b$

Case 2:- If f_1 and f_2 are linearly independent on $a \le x \le b$
Let $W(x)$ = Wronskian of f_1 and f_2

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

Diff. both side w.r.t to x, we get

$$W'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ f_1''(x) & f_2''(x) \end{vmatrix}$$

$$= \begin{vmatrix} f_1(x) & f_2(x) \\ f_1''(x) & f_2''(x) \end{vmatrix} \quad -(2)$$

As f_1 and f_2 are solution of

$a_0(x) f_1''(x) + a_1(x) f_1'(x) + a_2(x) f_1(x) = 0$ and

$a_0(x) f_2''(x) + a_1(x) f_2'(x) + a_2(x) f_2(x) = 0$

$$f_1''(x) = -\frac{a_1(x)}{a_0(x)} f_1'(x) - \frac{a_2(x)}{a_0(x)} f_1(x)$$

$$f_2''(x) = -\frac{a_1(x)}{a_0(x)} f_2'(x) - \frac{a_2(x)}{a_0(x)} f_2(x)$$

Subsitituting this value 2 in we get

$$W'(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ -\frac{a_1(x)}{a_0(x)} f_1'(x) - \frac{a_2(x)}{a_0(x)} f_1(x) & -\frac{a_1(x)}{a_0(x)} f_2'(x) - \frac{a_2(x)}{a_0(x)} f_2(x) \end{vmatrix}$$

$$= \begin{vmatrix} f_1(x) & f_2(x) \\ -\frac{a_1(x)}{a_0(x)} f_1'(x) & -\frac{a_1(x)}{a_0(x)} f_2'(x) \end{vmatrix} + \begin{vmatrix} f_1(x)\ 0 & 0 \\ -\frac{a_2(x)}{a_0(x)} f_1(x) & -\frac{a_2(x)}{a_0(x)} f_2(x) \end{vmatrix}$$

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$$W(x) = -\frac{a_1(x)}{a_0(x)} \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$W'(x) = -\frac{a_1(x)}{a_0(x)} W(x)$$

$$\frac{dW}{dx} + \frac{a_1(x)}{a_0(x)} W(x) = 0 \qquad a \le x \le b$$

Integrating factor: $e^{\int \frac{a_1(x)}{a_0(x)} dx}$

$$W \cdot e^{\int \frac{a_1(x)}{a_0(x)} dx} = c$$

$$W = c\, e^{-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}}$$ Integrating from x_0 to x, where is any arbitrary pt. of $[a, b]$

Let $x = x_0$ $\qquad c = W(x_0)$

$$W(x) = \boxed{W(x_0)} \cdot e^{-\int_{x}^{x_0} \frac{a_1(t)\,dt}{a_0(t)}}$$ for all x on $a \le x \le b$

Assume that $W(x_0) = 0$.

$\Rightarrow W(x_0) = 0$ for all $a \le x \le b$.

$\Rightarrow f_1$ and f_2 are linearly dependent which is a contradication.

$\therefore W(x_0) \neq 0$

$\Rightarrow W(x) = 0$ for

As x_0 is an arbitrary constant point of $a \le x \le b$, we conclude that $W(x) \neq 0$ for all $x \in [a, b]$

Theorem 2: let f_1 and f_2 be any two linearly independent soln. of diff. eqn.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad -(1)$$

The every solution f of (1) can be expressed as suitable linear combination of f_1 and f_2

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Proof:- Let x_0 be an arbitrary point of the interval $a \le x \le b$
Consider the following system of two linear algebraic equation in two unknown K_1 and K_2

$$\left.\begin{array}{l} K_1 f_1(x_0) + K_2 f_2(x_0) = f(x_0) \\ K_1 f_1'(x_0) + K_2 f_2'(x_0) = f'(x_0) \end{array}\right] -(2)$$

Since f_1 and f_2 are linearly independent on $a \le x \le b$
(Wronskian is zero for x in $a \le x \le b$).

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix} \ne 0 \qquad Ax = b$$
$$x = A^{-1} b.$$

$\Rightarrow$ The system of eq 2 has unique solution
let the solution be $K_1 = C_1$ and $K_2 = C_2$

$\Rightarrow C_1 f_1(x_0) + C_2 f_2(x_0) = f(x_0)$
($C_1 f_1$ (f and $C_1 f_1 + C_2 f_2$ have equal value at x_0)

$\Rightarrow C_1 f_1'(x_0) + C_2 f_2'(x_0) = f'(x_0)$.
(the value of the derivative of f and $C_1 f_1 + C_2 f_2$ are equal at x_0)

$$C_1 f_1(x_0) + C_2 f_2(x_0) - f(x_0) = 0$$
$$C_1 f_1'(x_0) + C_2 f_2'(x_0) - f'(x_0) = 0$$

$\Rightarrow C_1 f_1(x) + C_2 f_2(x) - f(x) = 0 \quad \forall x \in [a,b]$
$f(x) = C_1 f_1(x) + C_2 f_2(x)$ all for $x \in [a,b]$.
Hence f is expressed as linear comb. of f_1 and f_2

Types of linear system

General linear system of two first order diff. eqn in two unknown func. x and y is of the form

$$\left.\begin{array}{l} a_1(t)\dfrac{dx}{dt} + a_2(t)\dfrac{dy}{dt} + a_3(t)x + a_4(t)y = f_1(t) \\ b_1(t)\dfrac{dx}{dt} + b_2(t)\dfrac{dy}{dt} + b_3(t)x + b_4(t)y = f_2(t) \end{array}\right] -(1)$$

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Eg:- With constant coefficient

$$2\frac{dx}{dt} + 3\frac{dy}{dt} - 2x + y = t^2$$
$$\frac{dx}{dt} - 2\frac{dy}{dt} + 3x + 4y = e^t$$

A sol. is an order pair (f, g) . $x = f(t)$, $y = g(t)$ and satisfy (2) on some interval $a \le t \le b$

The general linear sol. of three first order diff. in three unknown function x, y and z is of the form:

$$a_1(t)\frac{dx}{dt} + a_2(t)\frac{dy}{dt} + a_3(t)\frac{dz}{dt} + a_4(t)x + a_5(t)y + a_6(t)z =$$
$$b_1(t)\frac{dx}{dt} + b_2(t)\frac{dy}{dt} + b_3(t)\frac{dz}{dt} + b_4(t)x + b_5(t)y + b_6(t)z$$
$$c_1(t)\frac{dx}{dt} + c_2(t)\frac{dy}{dt} + c_3(t)\frac{dz}{dt} + c_4(t)x + c_5(t)y + c_6(t)z =$$

(With constant co-efficient)

$$2\frac{dx}{dt} + 5\frac{dy}{dt} + 12\frac{dz}{dt} + x + 3y + 6z = t^2$$
$$\frac{dx}{dt} + 4\frac{dy}{dt} + 5\frac{dz}{dt} + 6x + 2y + 4z = t^4$$
$$3\frac{dx}{dt} + 7\frac{dy}{dt} + 8\frac{dz}{dt} + 2x + 5y + z = e^t$$

General form of linear system of two second order diff. eqn. in two unknown function x and y is system of the form.

$$a_1(t)\frac{d^2x}{dt^2} + a_2(t)\frac{d^2y}{dt^2} + a_3(t)\frac{dx}{dt} + a_4(t)\frac{dy}{dt} +$$
$$a_5(t)x + a_6(t)y = F_1(t)$$

$$b_1(t)\frac{d^2x}{dt^2} + b_2(t)\frac{d^2y}{dt^2} + b_3(t)\frac{dx}{dt} + b_4(t)\frac{dy}{dt} + b_5(t)x +$$
$$b_6(t)y = f_2(t)$$

Consider a special type of linear system of eqn:

$$\left.\begin{aligned}\frac{dx}{dt} &= a_{11}(t)x + a_{12}(t)y + f_1(t)\\ \frac{dy}{dt} &= a_{21}(t)x + a_{22}(t)y + f_2(t)\end{aligned}\right\} \text{Normal form}$$

Eg: $\frac{dy}{dt} = t^2x + (t+1)y + t^3$

$\frac{dx}{dt} = te^t x + t^3 y - e^t$

(with constant co-efficient)

$\frac{dx}{dt} = 2x + 7y + t^3$

$\frac{dy}{dt} = 5x + 11y - e^t$

The normal form of a linear system of three diff. eqn in three unknown x, y and z is.

$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + a_{13}(t)x + f_1(t)$

$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + a_{23}(t)x + f_2(t)$

$\frac{dx}{dt} = a_{31}(t)x + a_{32}(t)y + a_{33}(t)x + f_3(t)$

The normal form of a linear diff. eqn in n unknown $x_1, x_2, x_3 \ldots; x_n$.

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)$$

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = f(t)$$

(one unknown function)

$$\frac{dx}{dt} = \frac{dx_1}{dt} \qquad x_1 = x \qquad x_2 = \frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{dx_2}{dt} \qquad x_3 = \frac{d^2x}{dt^2} \qquad \cdots \qquad x_n = \frac{d^{n-1}x}{dt^{n-1}} \qquad \frac{d^n x}{dt^n} = \frac{dx_n}{dt}$$

$$\Rightarrow \frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3 \quad \cdots \quad \frac{dx_{n-1}}{dt} = x_n$$

$$x = x_1 \left[\frac{dx_n}{dt} = \frac{d^n x}{dt^n} = -a_1(t)\frac{d^{n-1}x}{dt^{n-1}} - a_2(t)\frac{d^{n-2}x}{dt^{n-2}} \cdots a_{n-1}(t)x_2 - a_n(t)x_1 + f(t) \right.$$

$$\frac{dx_n}{dt} = -a_1(t)x_n - a_2(t)x_{n-1} \cdots a_{n-1}(t)x_2 - a_n(t)x_1 + f(t)$$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3$$

$$\vdots$$

$$\frac{dx_{n-1}}{dt} = x_n$$

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$$\frac{dx_n}{dt} = -a_n(t)\,x_1 - a_{n-1}(t)\,x_2 \cdots a_2(t)\,x_{n-1} - a_1(t)\,x_n + f(t)$$

Differential Operation

The diff. operator $\frac{dx}{dt}$ is denoted by Dx

$$Dx = \frac{dx}{dt}$$

$$\Rightarrow D^2x = \frac{d^2x}{dt^2},\ D^3x = \frac{d^3x}{dt^3} \cdots\cdots D^n x = \frac{d^n x}{dt^n}$$

$$(D+a)x = Dx + ax = \frac{dx}{dt} + ax$$

$$(D^2+D)x = D^2x + Dx = \frac{d^2x}{dt} + \frac{dx}{dt}$$

$$(D^n + D^m)x = D^n x + D^m x = \frac{d^n x}{dt} + \frac{d^m x}{dt}$$

General n^{th} order diff. eqn. with constant co-efficient

$$a_0\frac{d^n x}{dt^n} + a_1\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}\frac{dx}{dt} + a_n x = F(t)$$

can be written as

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n)\,x = F(t)$$

Consider the diff. operator:

$$3D^2 + 5D - 2$$

$$(3D^2+5D-2)x = 3D^2x + 5Dx - 2x$$

$$= 3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 2x.$$

$$(3D^2+5D-2)\,t^3 = 3D^2(t^3) + 5D(t^3) - 2t^3$$

$$= 3(6t) + 5(3t^2) - 2t^3$$

$$= 18t + 15t^2 - 2t^3$$

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On operator for linear system with constant coeff

Consider a linear system of equation of the form

$$L_1 x + L_2 y = f_1(t)$$
$$L_3 x + L_4 y = f_2(t)$$

where L_1, L_2, L_3 and L_4 are linear diff. operator with constant co-efficients

$$L_1 = a_0 D^m + a_0 D^{m-1} + \dots + a_{m-1} D + a_m$$
$$L_2 = b_0 D^n + b_1 D^{n-1} + \dots + b_{n-1} D + b_n$$
$$L_3 = c_0 D^p + c_1 D^{p-1} + \dots + c_{p-1} D + c_p$$
$$L_4 = d_0 D^q + d_1 D^{q-1} + \dots + d_{q-1} D + d_q$$

Ex $2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$

$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2$

$(2D-3)x - 2Dy = t \quad -(1)$

$(2D+3)x + (2D+8)y = 2 \quad -(1)$

$L_1 = 2D-3 \qquad L_2 = 2D$

$L_3 = 2D+3 \qquad L_4 = 2D+8$

Applying $(2D+8)$ on eq 1 and $2D$ on eq 2

$(2D+8)(2D-3)x - 2Dy(2D+8) + (2D+3)(2D)x + 2Dy(2D+8) = t(2D+8) + 2(2D)$

$(4D^2 + 16D - 6D - 24)x + (4D^2 + 6D)x = 2\frac{d(t)}{dt} + 8t + 2(\dots)$

$[8D^2 + 16D - 24]x = 2 + 8t$

$[4D^2 + 8D - 12]x = 2(1+4t)$

Auxiliary eqn is

$4m^2 + 8m - 12 = 0$

$m^2 + 2m - 3 = 0$

$m^2+3m-m-3=0$

$m(m+3)-1(m+3)=0$

$(m-1)(m+3)=0$

$m=1,-3.$

$x_c(t)=c_1e^t+c_2e^{-3t}$

RHS of eqn 3 is $1+4t$

$1 \to \{1\}$

$4t \to \{t,1\}$

$\therefore$ UC set $\Rightarrow \{t,1\}$

$x_p(t)=At+B$ (where A and B are constant to be determined)

$x_p' = A \qquad x_p''=0.$

Substituting these value in eq(3) we get.

$(4\times 0+8A-12At-12B)=1+4t$

$8A-12B-12At=1+4t.$

$4t=-12At$

$A=-1/3.$

$-\frac{8}{3}-12B=1$

$-12B=\frac{11}{3}$

$B=-\frac{11}{36}$

$$\boxed{x_1(t)=c_1e^t+c_2e^{-3t}-\frac{t}{3}-\frac{11}{36}}$$

du, gy = Now applying $(2D+3)$ on eq 1 and $(2D-3)$ on eq 2

$(2D-3)(2D+3)x-2D(2D+3)y-(2D+3)(2D-3)x-(2D+8)(2D-3)y=(2D+3)t-(2D-3)2.$

$\Rightarrow(-4D^2-6D-4D^2+6D-16D+24)=2\frac{dt}{dt}+3t-\frac{2d(2)}{dt}+6$

$(-8D^2-16D+24)y=8+3t$

$(D^2+2D-3)y=-1-\frac{3t}{8} \quad -(4)$

Auxiliary eqn:

$m^2+2m-3=0$

$m^2+3m-m-3=0$

$m(m+3)-1(m+3)=0$

$(m-1)(m+3)=0$

$m=1,-3$

$y_c(t)=d_1e^t+d_2e^{-3t}$

RHS of eq 4 $= -1-\frac{3t}{8}$

$1 \rightarrow \{1\}$

$\frac{3t}{8} \rightarrow \{t,1\}$.

UC set $\rightarrow \{t,1\}$.

$y_p = At+B$ (where A and B const to be determined)

$y_p' = A$

$y_p'' = 0$

Substitute these value in eq 4 we get

$0+2A-3At-3B = -1-\frac{3t}{8}$.

$\Rightarrow 3At = \frac{3t}{8}$

$A = 1/8$.

$\frac{1}{4}-3B = -1$

$\Rightarrow 3B = +\frac{5}{4}$

$\boxed{B = \frac{5}{12}}$

$y(t) = d_1e^t+d_2e^{-3t}+\frac{t}{8}+\frac{5}{12}$

$(c_1, c_2, d_1, d_2) \rightarrow$ elimination?

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$(2D-3)x - 2Dy = t$

$(2D+3)x + (2D+8)y = 2$

$$\begin{vmatrix} 2D-3 & -2D \\ 2D+3 & 2D+8 \end{vmatrix} \Rightarrow 4D^2+16D-6D-24+4D^2+6D$$

$$8D^2+16D-24=0$$

Only two of the four ~~DE~~ constants c_1, c_2, d_1 and d_2 can be independent.

Now,

$(2D+3)x + (2D+8)y = 2.$

$2\frac{dx}{dt} + 3x + 2\frac{dy}{dt} + 8y = 2.$

Next

$(2D-3)x - 2Dy = t$

$2\frac{dx}{dt} - 3x - 2\frac{dy}{dt} = t.$

$$2\left[c_1e^t - 3c_2e^{-3t} - \frac{1}{3}\right] - 3\left[c_1e^t + c_2e^{-3t} - \frac{t}{3} - \frac{11}{36}\right]$$

$$-2\left[d_1e^t - 3d_2e^{-3t} + \frac{1}{8}\right] = t$$

$$\Rightarrow 2c_1e^t - 6c_2e^{-3t} - \frac{2}{3} - 3c_1e^t - 3c_2e^{-3t} + t + \frac{11}{12} - 2d_1e^t +$$

$$6d_2e^{-3t} - \frac{1}{4} = t$$

$$e^t(2c_1 - 3c_1 - 2d_1) + e^{-3t}(-6c_2 - 3c_2 + 6d_2) + t \cdot (1) + \left(-\frac{2}{3} - \frac{1}{4} + \frac{11}{12}\right) = t.$$

$$e^t(2c_1 - 3c_1 - 2d_1) + e^{-3t}(-6c_2 - 3c_2 + 6d_2) + 1 \cdot (t) = t$$

On comparing co-efficient

$-c_1 - 2d_1 = 0$

$-c_1 = 2d_1$

$d_1 = -\frac{c_1}{2}$

$-9c_2 + 6d_2 = 0$

$9c_2 = 6d_2$

$d_2 = \frac{3}{2}c_2.$

$$x(t) = \frac{2c_1 e^t}{3} + 2c_2 e^{-3t} - \frac{t}{3} - \frac{11}{36}$$
$$y(t) = \frac{c_1 e^t}{2} + \frac{3c_2}{2} e^{-3t} + \frac{t}{8} + \frac{5}{12}$$
→ Required Solution

Two equation in two unknown.

Normal

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + f_1(t)$$
$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + f_2(t)$$
— *

solution $(f(t), g(t))$

$a_{11}, a_{12}, a_{21}, a_{22}, f_1$ and f_2 are continuous on interval $[a, b]$

Eg:
$$\frac{dx}{dt} = 2x - y$$
$$(D-2)x + y = 0$$
$$\frac{dy}{dt} = 3x + 6y$$
$$(D-6)y - 3x = 0$$
— Homogenous.

$$\frac{dx}{dt} = 2x - y + t^2$$
$$\frac{dy}{dt} = 3x + 6y + e^t$$
→ Non homogenous

Theorem: Let the function $a_{11}, a_{12}, a_{21}, a_{22}, f_1$ and f_2 in system * are continous on the real interval $a < x < b$. Let t_0 be any pt. of the interval $a < t < b$ and let c_1 and c_2 be arbitrary constant. Then there exist a unique solution $x = f(t)$ and $y = g(t)$ of the system (*) such tha

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Theorem: Let $x = f_0(t)$ and $y = g_0(t)$ be the solution of non-homogeneous system (*) and let $x = f(t), y = g(t)$ be any soln. of the corresponding homogenous to (*)

Then $x = f(t) + f_0(t)$ and $y = g(t) + g_0(t)$

Homog. Non-homog homog. Non homog

is a solution of * (Non-homogenous)

Theorem:

$x = f_1(t)$ $y = g_1(t)$] linearly independent
$x = f_2(t)$ $y = g_2(t)$] solution of homogenous to (*)

Then general solution of (*) (Non-homog.) is given by

$x = c_1 f_1(t) + c_2 f_2(t) + f_0(t)$] where c_1 and c_2 are
$y = c_1 g_1(t) + c_2 g_2(t) + g_0 t$] arbitrary constant

Unit - 3.

Two equation in two unknown function

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1 x + b_1 y \\ \frac{dy}{dt} &= a_2 x + b_2 y\end{aligned}\right] -(1) \qquad A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

$$\begin{vmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{vmatrix} = 0.$$

$$(a_1 - \lambda)(b_2 - \lambda) - a_2 b_1 = 0$$

$$a_1 b_2 - a_1 \lambda - b_2 \lambda + \lambda^2 - a_2 b_1 = 0$$

$$\lambda^2 - (a_1 + b_2)\lambda + (a_1 b_2 - a_2 b_1) = 0 \Rightarrow \text{Characterstic equation.}$$

2 Roots → Real and distinct, Conjugate, Real and equal

Case I: Real and distinct root

λ_1, λ_2.

$$x = A_1 e^{\lambda_1 t} \qquad x = A_2 e^{\lambda_2 t}$$
$$y = B_1 e^{\lambda_1 t} \qquad y = B_2 e^{\lambda_2 t}$$

{where A_1, A_2, B_1 and B_2 are definite constant}

The general solution

$$x = C_1 A_1 e^{\lambda_1 t} + C_2 A_2 e^{\lambda_2 t}$$
$$y = C_1 B_1 e^{\lambda_1 t} + C_2 B_2 e^{\lambda_2 t}$$

{where C_1 and C_2 are arbitary constant}

Ex: $\frac{dx}{dt} = 6x - 3y$ $\qquad A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$.

$\frac{dy}{dt} = 2x + y$

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Putting $|A-\lambda I|=0$

$$\begin{vmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(1-\lambda)+6=0$$

$$6-7\lambda+\lambda^2+6=0$$

$$\lambda^2-7\lambda+12=0$$

$$\lambda^2-4\lambda-3\lambda+12=0$$

$$\lambda(\lambda-4)-3(\lambda-4)=0$$

$$\lambda=3,\ 4.$$

$$\lambda_1=3,\ \lambda_2=4$$

Putting $(A-\lambda I)\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3A=3B$$

$$A=B.$$

$$A=1,\ B=1.$$

$$x=e^{3t},\ y=e^{3t}$$

Putting $(A-\lambda_2 I)\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2A-3B=0$$

$$2A=3B$$

$$A=3,\ B=2$$

$$x=3e^{4t},\ y=2e^{4t}$$

General solution:

$x(t)=3c_2e^{4t}+c_1e^{3t}$

$y(t)=c_1e^{3t}+2c_2e^{4t}$

$\left\{ \begin{array}{l} \text{where } c_1, c_2 \text{ are} \\ \text{arbitrary const.} \end{array} \right\}$

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Case 2: Roots are conjugate complex:
Let $r_1 = a + ib$, $r_2 = a - ib$ be the roots of characteristic eqn of 1.

Then the two linearly independent soln are
$x = e^{at}[A_1 \cos bt - A_2 \sin bt]$
$y = e^{at}[A_1 \cos bt + A_1 \sin bt]$

$x = e^{at}[B_1 \cos bt - B_2 \sin bt]$
$y = e^{at}[B_2 \cos bt + B_1 \sin bt]$
[where B_1, B_2, A_1 and A_2 are definite constant]

$x = e^{at}[C_1(A_1 \cos bt - A_2 \sin bt) + C_2(A_2 \cos bt + A_1 \sin bt)]$
$y = e^{at}[C_1(A_1 \cos bt - B_2 \sin bt) + C_2(A_2 \cos bt + B_1 \sin bt)]$

Ex: $\frac{dx}{dt} = 3x + 2y$ $A = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$,

$\frac{dy}{dt} = -5x + y$

Putting $|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} = 0$

$(3-\lambda)(1-\lambda) + 10 = 0$
$3 - 4\lambda + \lambda^2 + 10 = 0$
$\lambda^2 - 4\lambda + 13 = 0$
$D = \cancel{-16} + 16 - 52 = -36$
$\sqrt{D} = \pm 6i$
Roots are $\frac{4 \pm 6i}{2} = 2 \pm 3i$

$x = Ae^{(2+3i)t} = 2[e^{2t} \cdot e^{i3t}]$
$2e^{2t}[\cos 3t + i \sin$

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$\lambda_1 = 2+3i$, $\lambda_2 = 2-3i$

$$\begin{bmatrix} 1-3i & 2 \\ -5 & -1-3i \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-3i)A + 2B = 0$

$(1-3i)A = -2B$

$A = 2, \quad B = -1+3i$

$x = Ae^{(2+3i)t}$

$= 2[e^{2t} \cdot e^{3it}]$

$= 2e^{2t}[\cos 3t + i\sin 3t]$

$= e^{2t}[2\cos 3t + i(2\sin 3t)]$

$y = Be^{(2+3i)t}$

$(-1+3i)[e^{2t} \cdot e^{3it}]$

$e^{2t}(-1+3i)[\cos 3t + i\sin 3t]$

$e^{2t}[-\cos 3t - i\sin 3t + 3i\cos 3t - 3\sin 3t]$

$e^{2t}[(-\cos 3t - 3\sin 3t) + i(3\cos 3t - \sin 3t)]$

$x = 2e^{2t}\cos 3t$ $\qquad$ $y = e^{2t}(-\cos 3t - 3\sin 3t)$

$x = 2e^{2t}\sin 3t$ $\qquad$ $y = e^{2t}(3\cos 3t - \sin 3t)$

General solution

$x = 2e^{2t}[c_1\cos 3t + c_2\sin 3t]$

$y = e^{2t}[c_1(-\cos 3t - 3\sin 3t) + c_2(3\cos 3t - \sin 3t)]$

Case 3: Real and equal roots:

$\lambda_1 = \lambda_2 = \lambda$

$x = Ae^{\lambda t}$ $\qquad$ $x = (A_1t + A_2)e^{\lambda t}$

$y = Be^{\lambda t}$ $\qquad$ $y = (B_1t + B_2)e^{\lambda t}$

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where A_1, B_1, A_2, B_2, A and B are definite constant}

General soln

$x = C_1 A e^{\lambda t} + C_2 (A_1 t + A_2) e^{\lambda t}$
$y = C_1 B e^{\lambda t} + C_2 (B_1 t + B_2) e^{\lambda t}$
{where C_1 and C_2 are arbitrary const}

Ex: $\frac{dx}{dt} = 4x - y$ $\qquad A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

$\frac{dy}{dt} = x + 2y$

Putting $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$8 - 6\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

$\lambda = 3, 3.$

$(A - \lambda I) \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A - B = 0 \Rightarrow \boxed{A = B}$

Assume $A = B = 1$

$x = e^{3t}$ $\qquad y = e^{3t}.$

$x = (A_1 t + A_2) e^{3t}$ $\qquad y = (B_1 t + B_2) e^{3t}$

$\frac{dx}{dt} = 3e^{3t}(A_1 t + A_2) + A_1 e^{3t}$

$\frac{dy}{dt} = 3e^{3t}(B_1 t + B_2) + B_1 e^{3t}.$

$\frac{dx}{dt} = 4x - y$

Putting the value

$A_1 e^{3t} + 3(A_1 t + A_2) e^{3t} = 4(A_1 t + A_2) e^{3t} - (B_1 t + B_2) e^{3t}$

$y = (B_1 t + B_2) e^{3t}$

Putting the value of $\frac{dy}{dt}$

$\frac{dy}{dt} = x + 2y$

$e^{3t} B_1 + 3 e^{3t} (B_1 t + B_2) = (A_1 t + A_2) e^{3t} + (2B_1 t + 2B_2) e^{3t}$

$(B_1 + 3B_2 - A_2 - 2B_2) e^{3t} + (3B_1 - A_1 - 2B_1) te^{t} = 0$

$A_1 - A_2 + B_2 = 0$

$B_1 - A_2 + B_2 = 0$

$- \quad + \quad -$

$A_1 - B_1 = 0$

$\boxed{A_1 = B_1}$

Let $A_1 = B_1 = 1$

$A_1 - A_2 + B_2 = 0$

$1 - A_2 + B_2 = 0$

$B_2 = A_2 - 1$

Let $A_2 = 1$, $B_2 = 0$

$x = (A_1 t + A_2) e^{3t} = (t+1) e^{3t}$

$y = (B_1 t + B_2) e^{3t} = te^{3t}$

Hence general solution is

$x = C_1 e^{3t} + C_2 (t+1) e^{3t}$

$y = C_1 e^{3t} + C_2 t(e^{3t})$

where C_1 and C_2 are arbitrary const.

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Simultaneous eqn. of the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$, P, Q R ⇒ func. of x, y and z

direction cosine of the tangent to the curve.

1. $\frac{xdx}{y^2 z} = \frac{xdy}{xz} = \frac{dz}{y^2}$

$\frac{xdx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}}$

$x^2 dx = y^2 dy$

$\cancel{\frac{x^2}{2}} \cancel{+ y}$

Integerate both side

$\frac{x^3}{3} = \frac{y^3}{3} + c$

$x^3 - y^3 = 3c$.

$x^3 - y^3 = c_1$

Now consider first and third fraction

$\frac{xdx}{y^2 \cancel{z}} = \frac{dz}{y^2}$

$xdx = zdz$

Integrating both side

$\frac{x^2}{2} = \frac{z^2}{2} + c$

$x^2 - z^2 = 2c$

$x^2 - z^2 = c_2$.

Hence required soln. is

$x^3 - y^3 = c_1$

$x^2 - z^2 = c_2$